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SOME PROBLEMS ON ADDITION CHAINS

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled SOME PROBLEMS ON ADDITION-CHAINS submitted by KEVIN R. HEBB in partial fulfilment of the requirements for the degree of Master of Science.

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Date July 15/74

In Chapter III we study the difference $f(n) - f(m)$. Proofs of $f(mn) - f(n) = 0$ for all $n \neq 2^k$ and infinitely many n and $f(mn) - f(n)$ for all m and infinitely many n are given. Estimates of lim sup and lim inf of $f(n) - f(m)$ are then obtained. We conclude the chapter by proving that the equations $f(n) = f(3n)$ and $f(n) = f(3n) - 1$ each have infinitely many solutions.

The final chapter indicates directions for future research.

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INTRODUCTION

We say that a sequence of integers $1 = a_0 < a_1 < a_2 < \ldots < a_r = n$ is an addition chain for n if every element a_k can be written as the sum $a_i + a_j$ of two preceding elements of the sequence ($i = j$ is permitted). By $\ell(n)$ one means the smallest r for which there is an addition chain for n. What is the least number of multiplications needed to raise x to the n^{th} power? This question which arises in the study of "optimal methods of computation" suggests an investigation of the properties of $\ell(n)$.

In 1937 Arnold Scholz [12] published several inequalities for $\ell(n)$ in the form of problems (see (1.1.3) to (1.1.6)). Attempts to prove these have given rise to many more conjectures, a few of which we now mention (see Chapter IV for a complete list of conjectures). It was conjectured by W.R. Utz [20] that $\ell(2n) = \ell(n) + 1$ because it seemed unlikely that there could be any shorter chain for $2n$ than to double the last element of the shortest chain for n. An examination of a table of values of $\ell(n)$ (see Appendix I) shows that 191 is the smallest value for which $\ell(2n) \neq \ell(n) + 1$. For a list of all $n < 18270$ with $\ell(2n) \neq \ell(n) + 1$ see Appendix II. Edward Thurber [15] found an infinite set for which $\ell(n) = \ell(2n)$. Therefore the conjecture was changed to $\ell(n) \leq \ell(2n)$. The more general conjecture that $\ell(n) \leq \ell(mn)$ was shown to be false for some m when Robert Giese [5] proved that $\ell(3n) < \ell(n)$ for infinitely many n. Edward Thurber [18] improved this by showing: for each integer $b \geq 0$ there exists infinitely many n such that $\ell(mn) \leq \ell(n) - b$ if $m = 2^{2k+1} + 1$;

$k = 0,1,2,\ldots$. Chapter III begins with a proof of $\ell(5n) \leq \ell(n) - b$ for infinitely many n (as above, b is an arbitrary integer ≥ 0). We then generalize to obtain a proof of $\ell(mn) \leq \ell(n) - b$ for infinitely many n if $m \neq 2^k$; $k = 1,2,3,\ldots$. From this we conclude that $\overline{\lim_{n\to\infty}} (\ell(n)-\ell(mn)) = +\infty$ for all $m \neq 2^k$. Next we show that $\ell(mn) > \ell(n)$ for infinitely many n and all m . Hence $\underline{\lim_{n\to\infty}} (\ell(n)-\ell(mn)) < 0$ for all m . We conclude the chapter by showing that $\ell(n) = \ell(3n) - 1$ and $\ell(n) = \ell(3n)$ each have an infinite number of solutions.

Probably the most famous problem about addition chains which is still unsettled is the Scholz-Brauer conjecture, which states that $\ell(2^n-1) \leq n - 1 + \ell(n)$. Computer calculations combined with theorems [10] [17] show, in fact, that $\ell(2^n-1) = n - 1 + \ell(n)$ for $1 \leq n \leq 18$ and for $n = 20$, 24, and 32 . An ℓ^*-chain is one in which the condition $a_k = a_i + a_j$ $(0 \leq j \leq i \leq k-1)$ is replaced by $a_k = a_{k-1} + a_j$ $(0 \leq j \leq k-1)$. Alfred Brauer [2] in 1939 proved that $\ell(2^n-1) \leq \ell^*(2^n-1) \leq n - 1 + \ell^*(n)$ where $\ell^*(n)$ denotes the minimum length of an ℓ^*-chain. But in 1958 Walter Hansen [7] proved the remarkable theorem that for certain large values of n, $\ell(n) < \ell^*(n)$; thus, the Scholz-Brauer conjecture is not proved by arguing that among the chains of shortest length there is an ℓ^*-chain. Hansen's theorems (see (1.2.2) to (1.2.5)) showed that there was infinitely many n satisfying $\ell(n) < \ell^*(n)$; but the smallest n obtainable from his theorems had more than 18,000 digits (see (1.4.6)). Donald Knuth's computer calculations [11] showed that the smallest n for which $\ell(n) \neq \ell^*(n)$ was actually much smaller; in fact, $\ell^*(n) = \ell(n) + 1$

for $n = 12509, 13207, 13705, 15473$ and 16537. This list contains all $n < 18270$ for which $\ell(n) \neq \ell^*(n)$.

W.R. Utz [20] attacked the Scholz-Brauer conjecture in a different manner. His method depended upon determining the complete behaviour of $\ell(n)$ when $n = 2^{c_0}$ and $n = 2^{c_0} + 2^{c_1}$, $c_0 > c_1 \geq 0$ (see (1.1.16)). This method was extended to the case $n = 2^{c_0} + 2^{c_1} + 2^{c_2}$, $c_0 > c_1 > c_2 \geq 0$, by A.A. Gioia, M.V. Subbarao and M. Suggunamma [6]. The case $n = 2^{c_0} + 2^{c_1} + 2^{c_2} + 2^{c_3}$, which was settled by Donald Knuth [10], had added complications since $\ell(2^{c_0}+2^{c_1}+2^{c_2}+2^{c_3}) = c_0 + 2$ or $c_0 + 3$ depending upon the relationship among the c_i's. The complete behaviour of $\ell(2^{c_0}+2^{c_1}+2^{c_2}+2^{c_3}+2^{c_4})$ has not been determined yet, but it is known that $\ell(2^{c_0}+\ldots+2^{c_4}) = c_0 + 3$ or $c_0 + 4$ and that both occur since $\ell(31) = 7$ and $\ell(789) = 13$. We determine some of the cases when $\ell(2^{c_0}+\ldots+2^{c_4}) = c_0 + 3$ (see Theorem (2.3.4)) but our list is far from complete (see Remark (2.3.5)).

In his book "The Art of Computer Programming", Vol. II, Donald Knuth introduces two new functions concerning addition chains and tabulates the known values of each. Let $c(r)$ be the smallest n such that $\ell(n) = r$ (see (2.3.3)). In Chapter II we show that $\log_2 c(r)$ is asymptotic to r. Let $d(r)$ be the number of solutions of the equation $\ell(n) = r$ (see (2.4.19)). In section (2.4) we prove that $\frac{1}{6}(2r^3-24r^2+118r-180) \leq d(r) \leq 2^{r-2} + (r-2)^2 - c(r)$ if $r \geq 7$ and use this to estimate the ratio $d(r)/d(r-1)$. We also prove that $\log_2 d(r)$ is asymptotic to r under the assumption that $d(r)$ is monotonic increasing. Although it seems obvious that $d(r)$ is an increasing function of r no proof of this has been found.

Paul Erdös [4] proved that $\ell(n) = \frac{\log n}{\log 2} + \frac{\log n}{\log \log n} + o(\frac{\log n}{\log \log n})$ for almost all n, thus improving Alfred Brauer's [2] result $\ell(n) \sim \frac{\log n}{\log 2}$.

In section (2.2) we show that each of the diophantine equations $x\ell(x) = y\ell(y)$ and $\ell(x)/x = \ell(y)/y$ have an infinite number of solutions (x,y). The equations one obtains when ℓ is replaced by ℓ^* also have an infinite number of solutions. We have not been able to determine what happens when these equations are extended to three variables (i.e., $\ell(x)/x = \ell(y)/y = \ell(z)/z$, $x\ell(x) = y\ell(y) = z\ell(z)$) but we do know that for a given x_0 there are only finitely many y such that $\ell(x_0)/x_0 = \ell(y)/y$.

It is easy to see that the Scholz-Brauer conjecture implies $\ell(2^{m+k}-2^k) \leq m + k + \ell(m) - 1$. M.V. Subbarao would like to know if $\ell(2^{m+1}-2) \leq m + \ell(m)$ implies the Scholz-Brauer conjecture.

CHAPTER I

SURVEY OF KNOWN RESULTS

§1.1. Origin of the Scholz-Brauer Conjecture. A sequence of integers

$$1 = a_0, a_1, a_2, \ldots, a_r = n \tag{1.1.1}$$

is called an addition chain for the positive integer n if for all $i = 1,2,\ldots,r$ we have

$$a_i = a_j + a_k, \quad 0 \leq k \leq j \leq i - 1 . \tag{1.1.2}$$

One may assume without any loss of generality that an addition chain ascending for if any two a_i's are equal, one of them may be dropped; and we can also rearrange the sequence (1.1.1) into ascending order and remove terms greater than n without destroying the addition chain property (1.1.2).

The phrase "addition chain (additionsketten) for n" was introduced by Arnold Scholz [12]. The integer r is called the length of the addition chain for n and the smallest possible value of r is denoted by $\ell(n)$. The sequences 1, 2, 4, 5, 8, 13; 1, 2, 4, 8, 12, 13; 1, 2, 3, 6, 12, 13; 1, 2, 3, 5, 7, 9, 11, 13; 1, 2, 3, 5, 7, 10, 13 are addition chains for 13 of length 5, 5, 5, 7 and 6 respectively. A few calculations will convince us that there is no chain of length 4 for the number 13, therefore $\ell(13) = 5$. It is easy to find a chain of length 11 for 191 but it is very difficult to prove by hand

that $\ell(191) > 10$.

In 1937 Arnold Scholz [12] published the following inequalities for $\ell(n)$ in the form of problems:

(1.1.3) $\quad m + 1 \leq \ell(n) \leq 2m$ for $2^m + 1 \leq n \leq 2^{m+1}$, $m \geq 1$

(1.1.4) $\quad \ell(ab) \leq \ell(a) + \ell(b)$

(1.1.5) $\quad \ell(2^n - 1) \leq n - 1 + \ell(n)$

(1.1.6) $\quad 1 \leq \overline{\lim_{n\to\infty}} \frac{\log 2}{\log n} \ell(n) \leq 2$.

He further raised the question of whether or not (1.1.3) and (1.1.6) can be improved.

Two years later Alfred T. Brauer [2] proved (1.1.3) and (1.1.4). His main result was the following:

(1.1.7) Theorem. If r is any positive and s any not negative integer, then $\ell(n) \leq (r+1)s + 2^r - 2$ for $2^{rs} \leq n < 2^{r(s+1)}$.

From Theorem (1.1.7) it follows that

(1.1.8) $\quad \ell(n) \leq \min_{1\leq r\leq m} \{(1 + \frac{1}{r}) \frac{\log n}{\log 2} + 2^r - 2\}$.

where $2^m \leq n < 2^{m+1}$. Brauer then set $r = [\log \log n] + 1$ for $n \geq 3$ and obtained

(1.1.9) $\quad \ell(n) < \frac{\log n}{\log 2}\{1 + \frac{1}{\log \log n} + \frac{2 \log 2}{(\log n)^{1-\log 2}}\}$

which proves

(1.1.10) Theorem. $\lim_{n\to\infty} \frac{\log 2}{\log n} \ell(n) = 1$.

Alfred Brauer could not decide whether or not (1.1.5) was always true. He did however define a special chain, which we will call a star chain following Donald Knuth [10], for which (1.1.5) is true. A star chain is one in which condition (1.1.2) is replaced by

(1.1.11) $$a_i = a_{i-1} + a_k, \quad k \leq i-1, \quad i = 1,2,\ldots,r .$$

That is, Brauer proved

(1.1.12) Theorem. $\ell(2^n-1) \leq \ell^*(2^n-1) \leq n - 1 + \ell^*(n)$, where $\ell^*(n)$ is the minimal length of all star chains for the integer n .

From now on we will refer to (1.1.5) as the Scholz-Brauer conjecture. Now, following Donald Knuth [10], we will introduce two auxiliary functions for convenience in our subsequent discussion:

(1.1.13) $\lambda(n) = [\log_2 n]$ = largest k such that $2^k \leq n$

(1.1.14) $v(n)$ = number of ones in the binary representation of n .

The next paper to appear was by W.R. Utz [20]. His main result was the following:

(1.1.15) Theorem. $\ell(2^n-1) \leq n - 1 + \ell(n)$ for all n with $v(n) = 1$ or 2 .

Utz's method of proof depended upon determining $\ell(n)$ for all n with $\nu(n) = 1$ or 2 ; that is, he proved

(1.1.16) Theorem. $\ell(n) = \lambda(n)$ whenever $\nu(n) = 1$ and $\ell(n) = \lambda(n) + 1$ whenever $\nu(n) = 2$.

Since any positive integer n can be written

$$(1.1.17) \qquad n = 2^{c_0} + 2^{c_1} + \ldots + 2^{c_t}, \; c_0 > c_1 > \ldots > c_t \geq 0$$

Utz thought his method might extend to arbitrary n but he was unable to carry this through. Utz also asked the following: Does the inequality

$$(1.1.18) \qquad \ell(n) < \ell(2n)$$

hold for all n ? (1.1.4) shows that $\ell(2n) \leq \ell(n) + 1$. Let $d(r)$ denote the number of solutions of the equation $\ell(n) = r$. Is $d(r) < d(r+1)$ for all $r \geq 1$?

§1.2. $\ell(n) < \ell^*(n)$. For several years optimistic people thought $\ell(n) = \ell^*(n)$; given an addition chain for n of minimal length $\ell(n)$ it is hard to believe that one cannot find a star chain for n of the same length. But in 1958 Walter Hansen [7] proved the remarkable theorem that

$$(1.2.1) \qquad \ell(n) < \ell^*(n)$$

for certain large values of n . The first of the following four theorems due to Hansen is the key to this sensational result.

(1.2.2) Theorem. If $n = 2^{c_0} + 2^{c_1} + \ldots + 2^{c_t}$, $t \geq 2$, $c_0 > 2c_1 + 16^t$, $c_i \geq c_{i+1} + 2 \cdot 16^t$ for $i = 1,2,\ldots,t-1$ then $\ell^*(n) = c_0 + t$.

(1.2.3) Theorem. For each positive integer D there exists an integer n such that

$$\ell^*(n) - \ell(n) > D(\ell(n) - [\frac{\log n}{\log 2}]) .$$

(1.2.4) Theorem. For each positive integer D there exists an integer n such that

$$\ell^*(n) - \ell^\circ(n) > D(\ell^\circ(n) - [\frac{\log n}{\log 2}]) .$$

(1.2.5) Theorem. $\ell^\circ(\frac{2^{mn}-1}{2^n-1}) \leq (m-1)n + \ell^\circ(m)$.

$\ell^\circ(n)$ is the minimal length of an ℓ°-chain for n . In an ℓ°-chain, certain of the elements are underlined; the condition is that $a_i = a_j + a_k$, where a_j is the largest underlined element less than a_i . The set of ℓ^*-chains is a subset of the ℓ°-chains. If $\ell^\circ(n) = \ell(n)$ for all n the Scholz-Brauer conjecture would be settled since Theorem (1.2.5) reduces to the Scholz-Brauer conjecture when $n = 1$. But it is not known whether or not $\ell(n) = \ell^\circ(n)$ for all n .

From (1.1.8) by choosing $r = [(1-\epsilon)\frac{\log \log n}{\log 2}]$ it follows that

(1.2.6) $$\ell(n) < \frac{\log n}{\log 2} + \frac{\log n}{\log \log n} + o(\frac{\log n}{\log \log n}) .$$

In 1960 Paul Erdös [4] showed that (1.2.6) was the best possible by proving the following:

(1.2.7) Theorem. For almost all n (i.e. for all n except a sequence of density zero)

$$\ell(n) = \frac{\log n}{\log 2} + \frac{\log n}{\log\log n} + o\left(\frac{\log n}{\log\log n}\right).$$

He also stated but did not prove

(1.2.8) $$\ell_r(n) = \frac{\log n}{\log r} + \frac{\log n}{(r-1)\log\log n} + o\left(\frac{\log n}{\log\log n}\right)$$

where $\ell_r(n)$ is the length of the shortest addition chain for n of order r. A sequence $1 = a_0 < a_1 < \ldots < a_k = n$ is said to be an addition chain of order r if each a_j is the sum of r or fewer a_i's where the indices do not exceed j.

§1.3. Gioia, Subbarao, Sugunamma. The next paper to attack the Scholz-Brauer conjecture was a joint effort by A.A. Gioia, M.V. Subbarao and M. Sugunamma [6]. They proved that the Scholz-Brauer conjecture held for all n with $v(n) = 3$ but they were unable to settle the conjecture when $v(n) = 4$ since they could not determine the complete behaviour of $\ell(n)$ when $v(n) = 4$. However, they proved the following theorems.

(1.3.1) Theorem. $\ell(n) = \lambda(n) + 2$ whenever $v(n) = 3$.

(1.3.2) Theorem. Let $n = 2^{c_0} + 2^{c_1} + 2^{c_2} + 2^{c_3}$, $c_0 > c_1 > c_2 > c_3 \geq 0$.

Then $c_0 + 2 \leq \ell(n) \leq c_0 + 3$ and $\ell(n) = c_0 + 2$ when any one of the following conditions is satisfied.

(A) $c_0 - c_1 = c_2 - c_3$

(B) $c_0 - c_1 = 3, \quad c_2 - c_3 = 1$

(C) $c_0 - c_1 = c_2 - c_3 + 1$.

If the Scholz-Brauer conjecture holds for n , it holds for $2n$ also, provided $\ell(2n) = \ell(n) + 1$; similarly, if the conjecture holds for n , it holds for $n + 1$, provided $\ell(n) + 1 = \ell(n+1)$. They therefore posed the problem of determining the general values of n for which $\ell(2n) = \ell(n) + 1$ and $\ell(n) + 1 = \ell(n+1)$.

Let $1 = a_0 < a_1 < a_2 < \ldots < a_r = n$ be a chain for n . We say it is a special chain of type B if there is an $m \leq r$ such that

(1.3.3)
$$a_i = a_{i-1} + a_j \qquad i = 1,2,\ldots,m$$
$$a_i = a_m + a_j \qquad i = m+1,\ldots,r \text{ for some } j < i .$$

Clearly if $m = r$, then the special chains of type B are just ℓ^*-chains. A.A. Gioia, M.V. Subbarao, and M. Sugunamma proved the following generalization of Brauer's Theorem (1.1.12).

(1.3.4) Theorem. Let m admit a shortest chain which is a special chain of type B. Let $n = 2^{c_1} + 2^{c_2} + \ldots + 2^{c_m}$ where $c_i - c_{i+1} = d$ for $i = 1,2,\ldots,m-1$; then $\ell(n) \leq d(m-1) + c_m + \ell(m)$.

In 1964, E.G. Straus [14] in reply to a question of Richard

Bellman [1], generalized the notion of an addition chain by asking the least number of multiplications needed to construct $a_1^{n_1}a_2^{n_2}\ldots a_k^{n_k}$. Following Brauer's proof of Theorem (1.1.7) Straus showed that

(1.3.5) Theorem. $\ell(n) \leq (r+1)s + 2^{kr} - k - 1$ if $2^{rs} \leq |n| < 2^{r(s+1)}$, $|n| = \max_{1\leq i\leq k}\{n_i\}$.

In 1965 A.M. Il'in [8], apparently unaware of the paper by Paul Erdös [4] showed that

$$(1.3.6) \qquad 0 < \overline{\lim_{n\to\infty}} \left(\frac{\ell(n)-\log_2 n}{\log_2 n}\right)\log_2\log_2 n \leq 1 .$$

He conjectured that this limit exists and equals one, which is what Paul Erdös had proved five years earlier. (Theorem (1.2.7)). In the same year C.T. Whyburn [21] pointed out how Brauer's proof of the Scholz-Brauer conjecture for star chains could be used to simplify the proof of the conjecture when $v(n) = 1, 2$ or 3.

In 1969 Kenneth B. Stolarsky [13] investigated the problem of calculated θ where

$$(1.3.7) \qquad \theta = \underline{\lim_{n\to\infty}} \left(\ell(2^n-1) - n\right)\frac{\log 2}{\log n} .$$

The upper bound $\theta \leq 1$ can be deduced from Brauer's Theorem (1.1.12) and by a detailed analysis of the structure of addition chains Kenneth Stolarsky obtained the lower bound $\theta > \frac{1}{3}$. If $\theta < 1$, the Scholz-Brauer conjecture follows immediately for infinitely many n. He says the fourth condition $c_0 - c_1 = 5$, $c_1 - c_2 = c_2 - c_3 = 1$ should be

added to Theorem (1.3.2) and conjectured that aside from these $v(n) = 4$ implies $\ell(n) = c_0 + 3$.

§1.4. Knuth's Calculations and Theorems. Donald Knuth's book "The Art of Computer Programming", volume II [10] consolidates the known results, presents some new proofs of old results, contains numerical tables of $\ell(n)$ and related functions (see (2.3.3), (2.4.19) and Appendix I and II) and determines the complete behaviour of $\ell(n)$ when $v(n) = 4$. Before stating Knuth's theorems we give some of the notation he introduced. Step i of the chain (1.1.1) is called a doubling if $j = k = i - 1$ in (1.1.2), i.e. $a_i = 2a_{i-1}$. If j (but not necessarily k) equals $i - 1$ we will say that step i as a star step. Finally, we will say that step i is a small step if

$$\lambda(a_i) = \lambda(a_{i-1}) . \tag{1.4.1}$$

(1.4.2) Theorem. If the addition chain (1.1.1) includes d doublings and $t = r - d$ nondoublings then (F_i is the i^{th} Fibonacci number)

$$n \leq 2^{d-1} F_{t+3} .$$

(1.4.3) Corollary. If the addition chain (1.1.1) includes t non-doublings and s small steps then

$$s \leq t \leq 3.271s .$$

Using this corollary Knuth improved Hansen's Theorem (1.2.2) as follows:

(1.4.4) Theorem. Let $n = 2^{c_0} + 2^{c_1} + \ldots + 2^{c_t}$, $c_0 > c_1 > c_2 > \ldots > c_t \geq 0$. If $c_0 > 2c_1 + 3.271(t-1)$ and $c_{i-1} > c_i + 2m$ for $1 \leq i \leq t$, where $m = 2^{[3.271(t-1)]} - t$ then $\ell^*(n) = c_0 + t$.

A slight modification of the proof of Theorem (1.2.3) gives

(1.4.5) Theorem. $\ell(2^A xy) \leq A + v(x) + v(y) - 1$ if $\lambda(x) + \lambda(y) \leq A$.

These two theorems enable us to find values of n for which $\ell(n) < \ell^*(n)$. The smallest n, for which $\ell(n) < \ell^*(n)$, obtainable from these theorems is

(1.4.6) $n = 2^{6106} + xy$ where $x = 2^{1016} + 1$, $y = 2^{2032} + 1$.

(1.4.7) Theorem. If $v(n) \geq 4$ then $\ell(n) \geq \lambda(n) + 3$, except in the following circumstances when $c_0 > c_1 > c_2 > c_3 \geq 0$ and $\ell(2^{c_0}+2^{c_1}+2^{c_2}+2^{c_3})$ equals $c_0 + 2$

(A) $c_0 - c_1 = c_2 - c_3$ (example: $n = 15$)

(B) $c_0 - c_1 = 3$, $c_2 - c_3 = 1$ (example: $n = 39$)

(C) $c_0 - c_1 = c_2 - c_3 + 1$ (example: $n = 23$)

(D) $c_0 - c_1 = 5$, $c_1 - c_2 = c_2 - c_3 = 1$ (example: $n = 135$).

In Theorem (2.3.4) and Lemma (3.4.1) we list some cases when $\ell(2^{c_0}+2^{c_1}+2^{c_2}+2^{c_3}+2^{c_4}) = c_0 + 3$.

The results of the extensive computer calculations carried out by Donald Knuth are very interesting. He found 29 counterexamples to W.R. Utz's conjecture that $\ell(n) < \ell(2n)$ for all n, the first few

being 191, 701, 743 (see Appendix II for the complete list). He also found five numbers such that $\ell^*(n) = \ell(n) + 1$; they are, 12509, 13207, 13705, 15473, 16537 . These are the only numbers less than or equal to 18269 for which $\ell(n) \neq \ell^*(n)$. When we compare these numbers with (1.4.6) we see the huge gap between the theory and the real situation. He introduces the function $c(r)$ where $c(r)$ is the smallest value of n such that $\ell(n) = r$ and gives a table of value of $c(r)$ (see (2.3.3)). He conjectures that the function $d(r)$, the number of solutions of $\ell(n) = r$, is monotonic increasing and gives a table of values of $d(r)$ (see (2.4.19)).

In section (2.3) we prove

(1.4.8) $$\log_2 c(r) \sim r$$

and use this to obtain

(1.4.9) $$\log_2 d(r) \sim r$$

under the assumption that $d(r)$ is monotonic increasing. We also show that

(1.4.10) $$\frac{1}{6}(2r^3-24r^2+118r-180) \leq d(r) \leq 2^{r-2} + (r-2)^2 - c(r)$$

if $r \geq 7$ and use this to obtain an estimate of the ratio $d(r)/d(r-1)$.

§1.5. Recent Results. Kenneth B. Stolarsky [13] conjectured that

(1.5.1) $$\ell(n) \geq \lambda(n) + k + 1 \quad \text{if} \quad \nu(n) \geq 2^k + 1, \quad k = 0,1,2,\ldots .$$

The main result of Edward G. Thurber's thesis [15] settles this conjecture for $k = 3$. A proof of the spectacular result that $\ell(2n) = \ell(n)$ for

infinitely many n is also given. His thesis also contains the characterization of $\ell(n)$ for many classes of integers with five or more ones in their binary representation.

(1.5.2) Theorem. $\ell(n) \geq \lambda(n) + 4$ if $\nu(n) \geq 9$.

(1.5.3) Theorem. The set of integers n such that $n = 2^m(23) + 7$ where $m \geq 5$ is an infinite class of integers for which $\ell(2n) = \ell(n) = m + 8$.

(1.5.4) Theorem [16]. If $n = 2^m(19) + 1$ and $m \geq 8$, then $\ell(n) = \ell(n+b) = m + 7$ for all integers b such that $1 \leq b \leq 21$.

In 1972 Robert P. Giese [5] showed that Knuth's conjecture, $\ell(n) \leq \ell(mn)$, is false for some m by proving:

(1.5.5) Theorem. $\ell(n) > \ell(3n)$ for infinitely many n.

In 1973 Edward Thurber [18] obtained improvements of Theorem (1.1.7) and Theorem (1.5.5)

(1.5.6) Theorem. $\ell(n) \leq \lambda(n) + 2^{k-1} - (k-1) + [\lambda(n)/k]$ for all integers $k \geq 1$.

(1.5.7) Theorem. If $m = 2^{2k+1} + 1$ for any $k \geq 0$ and b is an arbitrary nonnegative integer, there exists infinitely many infinite classes of integers n for which $\ell(mn) \leq \ell(n) - b$.

(1.5.8) Theorem. $\ell(2^n-1) = n - 1 + \ell(n)$ if $n = 1,2,3,\ldots,17,18,20,24,32$.

Thurber used Theorem (1.5.6) to obtain a set of lower bounds for $c(r)$.

(1.5.9) Theorem. For $k \geq 1$, if $r = (k+1)t + 2^{k-1} - k + 2 + j$, then $c(r) > 2^{kt+j+1}$ for $j = 0,1,2,\ldots,k-1$ and $c(r) > 2^{kt+k}$ if $j = k$.

In Chapter III we improve Theorems (1.5.5) and (1.5.7).

(1.5.10) Theorem. For all integers $m > 1$, $m \neq 2^k$ for $k \geq 1$ and b an arbitrary nonnegative integer, there exists infinitely many integers n for which $\ell(mn) \leq \ell(n) - b$.

(1.5.11) Theorem.
$$\ell(2^k \cdot 11) = \ell(3 \cdot 2^k \cdot 11) - 1, \quad k \geq 0$$
$$\ell(2^k \cdot 43) = \ell(3 \cdot 2^k \cdot 43) - 1, \quad k \geq 0$$
$$\ell(2^k \cdot 171) = \ell(3 \cdot 2^k \cdot 171), \quad k \geq 0$$
$$\ell(2^k \cdot 683) = \ell(3 \cdot 2^k \cdot 683), \quad k \geq 0 .$$

Conjecture (1.5.1) which can be written as

(1.5.12) $$\ell(n) \leq \lambda(n) + \log_2 v(n)$$

is still an open question. Recently Anne Cottrell [3], following Kenneth Stolarsky's method, has proven

(1.5.13) Theorem. $\ell(n) \geq \lambda(n) + \log_3 \nu(n)$.

In section (2.2) we prove

(1.5.14) Theorem. There exists an infinite number of solutions (x,y) to each of the diophantine equations $x\ell(x) = y\ell(y)$ and $\ell(x)/x = \ell(y)/y$.

The equations obtained upon replacing ℓ by ℓ^* also have an infinite number of solutions. We are unable to extend these chains of equations to three or more variables.

CHAPTER II

SOME PROPERTIES OF $\ell(n)$, $\ell^*(n)$, $c(r)$ AND $d(r)$

§2.1. Introduction. In the first part of this chapter we prove that there exists an infinite number of solutions to each of the diophantine equations $\ell(x)/x = \ell(y)/y$; $x\ell(x) = y\ell(y)$. Each of the equations $\ell^*(x)/x = \ell^*(y)/y$; $x\ell^*(x) = y\ell^*(y)$ also have an infinite number of solutions (x,y). Next we prove that $\log_2 c(r)$ is asymptotic to r and characterize some n with $\ell(n) = \lambda(n) + 3$. Then we take a look at the function $d(r)$, where $d(r)$ is the number of solutions of the equation $\ell(n) = r$. We obtain the bounds $(r \geq 7)$

$$(2.1.1) \qquad \frac{1}{6}(2r^3-24r^2+118r-180) \leq d(r) \leq 2^{r-2} + (r-2)^2 - c(r)$$

and use this to estimate the ratio $d(r)/d(r-1)$. Under the assumption that $d(r)$ is an increasing function of r we prove that $\log_2 d(r)$ is asymptotic to r. At the end of the chapter we consider a special class of chains which are different from star chains and which satisfy the Scholz-Brauer conjecture.

§2.2 Diophantine Equations.

(2.2.1) Theorem. The diophantine equation $\ell(x)/x = \ell(y)/y$ has an infinite number of solutions (x,y).

Proof. Using Theorems (1.1.16) and (1.4.7) it is easy to show that $\ell(2^{2^k-1}+2^{m+k}) = 2^k$ and $\ell(2^{2^k-1}+2^{2^k-k-1}+2^{m+k}+2^m) = 2^k + 1$ if $k \geq 1$ and $0 \leq m \leq 2^k - k - 1$. Therefore

$$\frac{\ell(2^{2^k-1}+2^{m+k})}{2^k} = \frac{\ell(2^{2^k-1}+2^{2^k-k-1}+2^{m+k}+2^m)}{2^k+1} \tag{2.2.2}$$

Dividing by $2^k - k - 1$ we get $\ell(x)/x = \ell(y)/y$ where $x = 2^{2^k-1} + 2^{m+k}$ and $y = 2^{2^k-1} + 2^{2^k-k-1} + 2^{m+k} + 2^m$.

(2.2.3) Another Proof. The following identities could also be used to prove Theorem (2.2.1).

$$\frac{\ell(2^r+2^{r-k})}{r+1} = \frac{\ell(2^r+2^{r-k}+2^{r-p-k})}{r+2} \tag{2.2.4}$$

if $r = 2^{p+k} + 2^p - 1$, $p \geq 1$, $k \geq 0$.

$$\frac{\ell(2^r+2^{r-k}+2^{r-k-p})}{r+2} = \frac{\ell(2^r+2^{r-k}+2^{r-k-p}+2^{r-k-p-m})}{r+3} \tag{2.2.5}$$

if $r = 2^{m+p+k} + 2^{m+p} + 2^m - 2$, $m \geq 1$, $p \geq 1$, $k \geq 0$ and $k \neq m$, $k \neq m+1$, $k \neq 3$ when $m = 1$, $m \neq 1$ when $k = 3$, $k \neq 5$ when $p = m = 1$, $p = m \neq 1$ when $k = 5$.

(2.2.6) Remark. There are solutions to $\ell(x)/x = \ell(y)/y$ besides those given by (2.2.2), (2.2.4) and (2.2.5). For example, $\ell(30)/30 = \ell(35)/35$; $\ell(36)/36 = \ell(42)/42$; $\ell(198)/198 = \ell(220)/220$. However, they do not seem to fit into any pattern.

(2.2.7) Theorem. The diophantine equation $\ell^*(x)/x = \ell^*(y)/y$ has an infinite number of solutions (x,y) .

Proof. The proof is implicit in the proof of Theorem (2.2.1) since the minimal addition chain for $2^{2^k-1} + 2^{m+k}$ given by (all steps indicated by "..." represent doublings) $2^0, 2^1, 2^2, \ldots, 2^{m+k}, \ldots, 2^{2^k-1}, 2^{2^k-1}+2^{m+k}$ is in fact a star chain. A similar remark applies to the minimal chain for $2^{2^k-1} + 2^{2^k-k-1} + 2^{m+k} + 2^m$ given by $2^0, 2^1, 2^2, \ldots, 2^m, \ldots, 2^{m+k}, 2^{m+k}+2^m, 2^1(2^{m+k}+2^m), 2^2(2^{m+k}+2^m), \ldots,$ $2^{2^k-m-k-1}(2^{m+k}+2^m) = 2^{2^k-1} + 2^{2^k-k-1}, \; 2^{2^k-1} + 2^{2^k-k-1} + 2^{m+k} + 2^m$.

(2.2.8) Unsolved Problem. Does there exist an infinity of triplets (x,y,z) such that $\ell(x)/x = \ell(y)/y = \ell(z)/z$? Certainly, the same question can be asked when ℓ is replaced by ℓ^* .

One solution to both equations is $\ell(2)/2 = \ell(4)/4 = \ell(6)/6 = \frac{1}{2}$. No other solution is known to the author.

(2.2.9) Theorem. The diophantine equation $x\ell(x) = y\ell(y)$ has an infinite number of solutions (x,y) .

Proof. Let $k = 2^s + 2, \; s \geq 2$.

Using Theorems (1.1.16) and (1.3.1) we get

$$\ell(2^k+2^{k-s+2}) = k + 1 \quad \text{and} \quad \ell(2^k+2^{k-s+1}+2^{k-s}) = k + 2 .$$

Therefore

$$\ell(2^k+2^{k-s+2})/(k+1) = \ell(2^k+2^{k-s+1}+2^{k-s})/(k+2) .$$

Multiplying by $(k+1)(2^{k-s})(2^s+2^2)$ we obtain

(2.2.10) $(2^k+2^{k-s+2})\ell(2^k+2^{k-s+2}) = (2^k+2^{k-s+1}+2^{k-s})\ell(2^k+2^{k-s+1}+2^{k-s})$.

(2.2.11) Another Proof. The following identities could also be used to prove Theorem (2.2.9).

(2.2.12) $(2^r+2^{r-k}+2^{r-k-m})(r+2) = (2^r+2^{r-k}+2^{r-k-m-1}+2^{r-k-m-2})(r+3)$

if $r = 2^{k+m+2} + 2^{m+2} + 1$, $m \geq 1$, $k \geq 4$.

(2.2.13) $(2^r+2^{r-k})(r+1) = (2^r+2^{r-k-1}+2^{r-k-2}+2^{r-k-3})(r+3)$

if $r = 2^{k+4} + 2^3 + 2^2 + 2^0$, $k \geq 3$.

(2.2.14) Remark. (63,71) is a solution of $x\ell(x) = y\ell(y)$ which does not satisfy (2.2.10), (2.2.12) or (2.2.13).

(2.2.15) Theorem. The diophantine equation $x\ell^*(x) = y\ell^*(y)$ has an infinite number of solutions (x,y) .

Proof. The proof is the same as that of Theorem (2.2.9) because there are chains of minimal length for $2^k + 2^{k-s+2}$ and $2^k + 2^{k-s+1} + 2^{k-s}$ that are star chains.

(2.2.16) Unsolved Problem. Does there exist an infinity of triplets (x,y,z) such that $x\ell(x) = y\ell(y) = z\ell(z)$; $x\ell^*(x) = y\ell^*(y) = z\ell^*(z)$? No solutions to these equations are known to the author.

(2.2.17) Theorem. For a given x_0 there are only finitely many y so that $\ell(x_0)/x_0 = \ell(y)/y$.

Proof. For any x_0 we can choose k such that

$$\frac{2k}{2^k} < \frac{\ell(x_0)}{x_0} \leq \frac{2(k+1)}{2^{k+1}} .$$

Since $(2 \log_2 y)/y$ is monotonic decreasing and $\ell(y)/y \leq (2 \log_2 y)/y$ for all y we get

$$\frac{\ell(y)}{y} \leq \frac{2k}{2^k} \text{ for all } y \geq 2^k .$$

Therefore any $y \geq 2^k$ can not be a solution of $\ell(x_0)/x_0 = \ell(y)/y$.

(2.2.18) Remark. A similar argument shows that there are only finitely many y so that $x_0\ell(x_0) = y\ell(y)$ for a given x_0 .

(2.2.19) Unsolved Problem. One easily obtains g.ℓ.b. $\ell(y)/y = 0$ and l.u.b. $\ell(y)/y = \frac{2}{3}$ from $0 \leq \ell(y)/y \leq (2 \log_2 y)/y$ but the g.ℓ.b. and l.u.b. of $\ell(y)/y$ is unknown when $2^k \leq y < 2^{k+1}$; $k = 1,2,3,\ldots$. The inequality $0 \leq \ell(y)/y \leq (2 \log_2 y)/y$ also gives $\lim_{y\to\infty} \ell(y)/y = 0$.

§2.3. $\log_2 c(r) \sim r$, $\ell(n) = \lambda(n) + 3$. Very little is known about $c(r)$. Edward Thurber [18] used his upper bound for $\ell(n)$ (Theorem (1.5.6)) to obtain a set of lower bounds for $c(r)$ (Theorem (1.5.9)). But these lower bounds for $c(r)$ combined with the trivial upper bound 2^r are not sufficient to obtain the asymptotic behaviour of $c(r)$.

(2.3.1) Theorem. $\log_2 c(r) \sim r$.

Proof. $\ell(n) < (\log_2 n)(1+f(n))$ by (1.1.9) where $f(n) = 1/\log\log n + 2\log 2/(\log n)^{1-\log 2}$. Combining this with the trivial upper bound $2^k > c(k)$ for all $k \geq 1$ we get

$$2^y \geq c([y]+1) > n$$

where $y = (\log_2 n)(1+f(n))$. Taking logs and dividing by $[y] + 1$ we get

$$\frac{y}{[y]+1} \geq \frac{\log_2 c([y]+1)}{[y]+1} > \frac{\log_2 n}{[y]+1} > \frac{1}{1+f(n)+1/\log_2 n} .$$

Since $1/\log_2 n$ and $f(n) \to 0$ as $y \to \infty$ we get $\log_2 c(r) \sim r$.

(2.3.2) Unsolved Problem. Obtain estimates for

$$\frac{1}{x} \sum_{r \leq x} c(r) ,$$

$$\overline{\lim_{r\to\infty}}\, c(r)/c(r-1)$$

and

$$\underline{\lim_{r\to\infty}}\, c(r)/c(r-1) .$$

What is the asymptotic value of $c(r)$?

(2.3.3)

r	c(r)	c(r)/c(r-1)
1	2	
2	3	1.50
3	5	1.67
4	7	1.40
5	11	1.57
6	19	1.73
7	29	1.53
8	47	1.62
9	71	1.51
10	127	1.79
11	191	1.50
12	379	1.98
13	607	1.60
14	1087	1.79
15	1903	1.75
16	3583	1.88
17	6271	1.75
18	11231	1.79

(2.3.4) Theorem. If $A > B > C > D > E \geq 0$ and $y = 2^A + 2^B + 2^C + 2^D + 2^E$ then $\ell(y) = A + 3$ in the following circumstances :

Case I $A - B = C - D$

Case II $A - B = C - E$

Case III $A - B = C - D + 1$

Case IV $A - B = D - E$

Case V $A - C = D - E$

Case VI $A - B = 3,\ C - D = 1$

Case VII $A - B = 5,\ B - C = C - D = 1$

Case VIII $A - B = 6, \quad B - C = C - D = D - E = 1$

Case IX $B - C = D - E$.

Proof. We will only prove the first three cases since these are required in a later proof. The proof of the remaining cases are similar. Theorem (1.4.7) shows that $\ell(y) \geq A + 3$ for all y with $v(y) \geq 5$. To complete the proof we must show that $\ell(y) \leq A + 3$. We do this by constructing a chain of length $A + 3$.

Case I (all steps indicated by "..." represent doublings):
$2^0, 2^1, 2^2, \ldots, 2^E, \ldots, 2^D, \ldots, 2^C, 2^C+2^D, 2^1(2^C+2^D), \ldots, 2^{A-C}(2^C+2^D) = 2^A + 2^B,$
$2^A + 2^B + 2^C + 2^D, \; 2^A + 2^B + 2^C + 2^D + 2^E$.

Case II: $2^0, 2^1, \ldots, 2^E, \ldots, 2^D, \ldots, 2^C, 2^C+2^E, 2^1(2^C+2^E), \ldots, 2^{A-C}(2^C+2^E) =$
$2^A + 2^B, \; 2^A + 2^B + 2^C + 2^E, \; 2^A + 2^B + 2^C + 2^D + 2^E$.

Case III: $2^0, 2^1, \ldots, 2^E, \ldots, 2^D, \ldots, 2^C, 2^C+2^D, 2^{C+1}+2^D, 2^1(2^{C+1}+2^D), \ldots,$
$2^{A-C-1}(2^{C+1}+2^D) = 2^A + 2^B, \; 2^A + 2^B + 2^C + 2^D, \; 2^A + 2^B + 2^C + 2^D + 2^E$.

(2.3.5) Remark. $\ell(y) = \lambda(y) + 3$ or $\lambda(y) + 4$ for all y with $v(y) = 5$. Probably the most practical way to characterize all numbers with $v(y) = 5$ is by listing the cases when $\ell(y) = \lambda(y) + 4$ since the "majority" of the numbers with $v(y) = 5$ have chains of length $\lambda(y) + 3$. Of the first 126 numbers with $v(y) = 5$ only the number 465 has an addition chain of minimal length $\lambda(y) + 4$, i.e. $\ell(465) = 12$. Of the 126 numbers with $v(y) = 5$ between 2^9 and 2^{10} only four have $\ell(y) = \lambda(y) + 4 = 13$. Only twelve of the 210 numbers with $v(y) = 5$ between 2^{10} and 2^{11} have $\ell(y) = \lambda(y) + 4 = 14$.

(2.3.6) Unsolved Problem. Find all n such that $\ell(n) = \lambda(n) + 3$. Such n can have $v(n) = 4, 5, 6, 7$ or 8.

§2.4. Bounds for $d(r)$. Like $c(r)$, the behaviour of the function $d(r)$ is relatively unknown. It is very likely that $d(r)$ is monotonic increasing but the proof of this appears to be very difficult. To obtain our bounds for $d(r)$ we use the characterizations of $\ell(n)$ when $v(n) = 1, 2, 3, 4$ and 5 (see Theorems (1.1.16), (1.3.1), (1.4.7) and (2.3.4)). A comparison of the bounds we obtain with the table of values at the end of this section (2.4.19) shows that the behaviour of $d(r)$ is far from being determined.

(2.4.1) Definition. $d_i(r)$ equals the number of solutions to the equation $\ell(n) = r$ where $2^{r-i} < n \leq 2^{r-i+1}$, $i = 1,2,3,\ldots$.

(2.4.2) Remark. $$d(r) = \sum_{i=1}^{r-\lambda(c(r))} d_i(r).$$

(2.4.3) Lemma. $d_1(r) = r$.

Proof. $d_1(r)$ equals one $(\ell(2^r) = r)$ plus the number of solutions of the equation

$$\ell(2^{r-1}+2^{a_2}+2^{a_3}+\ldots+2^{a_m}) = r$$

where $r - 1 > a_2 > a_3 > \ldots > a_m \geq 0$. Since

$$\ell(2^{r-1}+2^{a_2}) = r$$

by Theorem (1.1.16) and

$$\ell(2^{r-1}+2^{a_2}+\ldots+2^{a_m}) \geq r+1$$

for all $m \geq 3$ by Theorems (1.3.1) and (1.4.7) we get $d_1(r) = r$.

(2.4.4) Lemma. $d_2(r) = r^2 - 5r + 3$ if $r = 6, 7$ or 8 and $d_2(r) = r^2 - 5r + 4$ if $r \geq 9$.

Proof. $d_2(r)$ equals the number of solutions of the equation

$$\ell(2^{r-2}+2^{a_2}+2^{a_3}+\ldots+2^{a_m}) = r$$

where $r-2 > a_2 > a_3 > \ldots > a_m \geq 0$. Using Theorems (1.1.16), (1.3.1) and (1.4.7) we see that $d_2(r)$ equals the total number of solutions of the following equations.

(2.4.5) $\ell(2^{r-2}+2^{a_2}+2^{a_3}) = r$

(2.4.6) $\ell(2^{r-2}+2^{a_2}+2^{a_3}+2^{a_4}) = r$ and $r-2-a_2 = a_3 - a_4$

(2.4.7) $\ell(2^{r-2}+2^{a_2}+2^{a_3}+2^{a_4}) = r$ and $r-2-a_2 = a_3 - a_4 + 1$

(2.4.8) $\ell(2^{r-2}+2^{a_2}+2^{a_3}+2^{a_4}) = r$ and $r-2-a_2 = 3$, $a_3 - a_4 = 1$

(2.4.9) $\ell(2^{r-2}+2^{a_2}+2^{a_3}+2^{a_4}) = r$ and $r-2-a_2 = 5$,

$a_2 - a_3 = a_3 - a_4 = 1$.

Equation (2.4.5) has $\binom{r-2}{2}$ solutions. Equation (2.4.6) has

$$(r-2) + (r-4) + (r-6) + (r-8) + \ldots + 6 + 4 + 2$$

solutions if r is even because we have $(r-2k)$ solutions when $a_2 = r - k - 1$, $k = 2,3,4,\ldots,(r-2)/2$. Equation (2.4.6) has

$$(r-2) + (r-4) + (r-6) + \ldots + 5 + 3 + 1$$

solutions if r is odd. Similarly equation (2.4.7) has

$$(r-3) + (r-5) + (r-7) + \ldots + 4 + 2$$

solutions when n is odd and

$$(r-3) + (r-5) + (r-7) + \ldots + 3 + 1$$

solutions when n is even. Equation (2.4.8) has $r - 4$ solutions. Equation (2.4.9) has 1 solution if $r \geq 9$ and zero otherwise Summing we get

$$\binom{r-2}{2} + \frac{(r-1)(r-2)}{2} + (r-4) + 1 = \binom{r-2}{2} + \binom{r}{2} - 2 = r^2 - 5r + 4$$

if $r \geq 9$. $d_2(r) = r^2 - 5r + 3$ when $r = 6$, 7 and 8 because equation (2.4.9) has no solution if $r < 9$.

(2.4.10) Lemma. $d_3(r) \geq \binom{r-4}{3} + \binom{r-3}{3} + 2 - \binom{r-3}{2}$ if $r \geq 10$.

Proof. $d_3(r)$ equals the number of solutions of

$$\ell(2^{r-3}+2^{a_2}+2^{a_3}+\ldots+2^{a_m}) = r \tag{2.4.11}$$

where $r - 3 > a_2 > a_3 > \ldots > a_m \geq 0$. If the correct relationship holds among the exponents in equation (2.4.11) we have solutions for $m = 4$, 5, 6, 7 and 8. To obtain our lower bound we will count all

the solutions of

$$\ell(2^{r-3}+2^{a_2}+2^{a_3}+2^{a_4}) = r \tag{2.4.12}$$

and some of the solutions of

$$\ell(2^{r-3}+2^{a_2}+2^{a_3}+2^{a_4}+2^{a_5}) = r \tag{2.4.13}$$

By the proof of the previous lemma we know that the equation

$$\ell(2^{r-3}+2^{a_2}+2^{a_3}+2^{a_4}) = r - 1$$

has $\binom{r-3}{2} - 2$ solutions. Since there are $\binom{r-3}{3}$ numbers of the form

$$2^{r-3} + 2^{a_2} + 2^{a_3} + 2^{a_4} ,$$

equation (2.4.12) has $\binom{r-3}{3} - \binom{r-3}{2} + 2$ solutions. Using the first and third cases of Theorem (2.3.4) one can easily show that equation (2.4.13) has at least $\binom{r-4}{3}$ solutions.

(2.4.14) Remark. If we use Case II of Theorem (2.3.4) to get some more solutions to equation (2.4.13) we run into difficulties because some of the solutions for this case coincide with solutions already counted. The bound we just obtained for $d_3(r)$ is asymptotic to $c_1 r^3$ (c_1 is a constant) while the true bound is probably asymptotic to $c_2 r^4$ (c_2 is a constant) (see Remark (2.3.5)).

(2.4.15) Theorem.

$$\frac{1}{6}(2r^3-24r^2+118r-180) \leq d(r) \leq 2^{r-2} + (r-2)^2 - c(r)$$

if $r \geq 7$.

Proof. Since

$$d(r) = \sum_{i=1}^{r-\lambda(c(r))} d_i(r)$$

we simply add the results of the previous three lemmas to get the lower bound. The upper bound follows from the facts: 2^r is the largest number n such that $\ell(n) = r$, the number of solutions of $\ell(n) = r$ for $2^{r-2} < n \leq 2^r$ is $d_1(r) + d_2(r) = (r-2)^2$ if $r \geq 9$, and $c(r)$ is the smallest number n such that $\ell(n) = r$. This proves the theorem for $r \geq 10$ and a few calculations will prove it for $r = 7$, 8 and 9.

(2.4.16) Corollary. If $r \geq 8$ then

$$\frac{2r^3 - 24r^2 + 118r - 180}{6[2^{r-3}+(r-3)^2-c(r-1)]} \leq \frac{d(r)}{d(r-1)} \leq \frac{6[2^{r-2}+(r-2)^2-c(r)]}{2(r-1)^3 - 24(r-1)^2 + 118(r-1) - 180}.$$

(2.4.17) Theorem. If $d(k) \geq d(k-1)$ for $k \geq 2$ then $\log_2 d(r) \sim r$.

Proof. From the definitions of $c(r)$ and $d(r)$ we get

$$2 + \sum_{m=1}^{r} d(m) \geq c(r+1)$$

for all $r \geq 1$. By assumption $d(r) \geq d(m)$ for all $m \leq r$. Thus $2 + rd(r) \geq c(r+1)$ for all $r \geq 1$. Using the trivial upper bound 2^r we get

$$2^r \geq d(r) \geq (c(r+1)-2)/r .$$

Taking logs and dividing by r yields

$$1 \geq \frac{\log_2 d(r)}{r} \geq \frac{\log_2 (c(r+1)-2)}{r} - \frac{\log_2 r}{r} .$$

Since $\log_2 r/r \to 0$ as $r \to \infty$ and $\log_2 c(r) \sim r$ (Theorem 2.3.1) the proof is complete.

(2.4.18) Remark. Many questions concerning the set of numbers $d(r)$ remain unanswered. As example we mention the estimation of

$$\frac{1}{x} \sum_{r \leq x} d(r) ,$$

$$\overline{\lim_{r \to \infty}}\ d(r)/d(r-1)$$

and

$$\underline{\lim_{r \to \infty}}\ d(r)/d(r-1) .$$

What is the asymptotic value of $d(r)$?

(2.4.19)

r	d(r)	d(r)/d(r-1)	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
1	1		1			
2	2	2.00	2			
3	3	1.50	3			
4	5	1.67	4	1		
5	9	1.80	5	4		
6	15	1.67	6	9		
7	26	1.73	7	17	2	
8	44	1.69	8	27	9	
9	78	1.77	9	40	29	
10	136	1.74	10	54	71	1
11	246	1.81	11	70	156	9
12	432	1.76	12	88	295	37
13	772	1.79	13	108		
14	1382	1.79	14	130		
15	2481	1.80	15	154		

§2.5. Quasi Star Chains. In this section we consider a special chain which we will call a quasi star chain and prove that the Scholz-Brauer conjecture is true for these chains. Although our quasi star chain is a special case of Hansen's ℓ°-chains (see Theorem (1.2.5)) we feel it may be of some importance because the question of how far an ℓ^*-chain has to be generalized before its minimal length equals $\ell(n)$ is of interest.

(2.5.1) Definition. Let $1 = a_0, a_1, a_2, \ldots, a_r = n$ be a chain for n. We will call step i a quasi star step if

$$a_i = a_{i-2} + a_k, \quad 0 \leq k \leq i-2 . \tag{2.5.2}$$

Two quasi star steps are said to be "separated" if they are of the form $a_i = a_{i-2} + a_j$ $(0 \leq j \leq i-2)$; $a_m = a_{m-2} + a_k$ $(0 \leq k \leq m-2)$ and $a_k \geq a_i$.

(2.5.3) Definition. A chain is called a quasi star chain if it has at least one quasi star step, the others being star steps; and if there are two or more quasi star steps, they are separated.

Computer calculations [11] show that 12509, 13207, 13705, 15473 and 16537 are the only numbers less than 18270 such that $\ell(n) \neq \ell^*(n)$; in fact $\ell^*(n) = \ell(n) + 1$ for these five numbers. It is remarkable that there are only 8 chains of length 17 for 16537, namely 1, 2, 4, 8, 9, 16, 32, 64, 128, 256, 512, (521 or 1024), 1033, 2066, 4132, 8264, (8273 or 16528), 16537; and 1, 2, 4, 8, 16, 17, 32, 64, 128, 256, 512, 1024, (1041 or 2048), 2065, 4130, 8260, (8277

or 16520), 16537 . All 8 of these chains are quasi star chains and thus illustrates the importance of this concept.

(2.5.4) Unsolved Problem. Since $\ell^*(16537) = 18 = \ell^{**}(16537) + 1$ we see that $\ell^{**}(n) \neq \ell^*(n)$ for some n ($\ell^{**}(n)$ is the minimal length of a quasi star chain for n), but it is not known whether or not $\ell(n) = \ell^{**}(n)$ for all n.

If $\ell(n) = \ell^{**}(n)$ for all n, the Scholz-Brauer conjecture would be settled because of the following:

(2.5.5) Theorem. $\ell^{**}(2^n-1) \leq n - 1 + \ell^{**}(n)$.

Proof. The following array is clearly a quasi star chain with only one quasi star step. By the definition of a quasi star chain we can simply repeat the construction in lines (2.5.6), (2.5.7) and (2.5.8) if we have more than one quasi star step.

$$2^{a_0} - 1, 2^1(2^{a_0}-1), 2^2(2^{a_0}-1), \ldots, 2^{a_1-a_0}(2^{a_0}-1)$$

$$2^{a_1} - 1, 2^1(2^{a_1}-1), 2^2(2^{a_1}-1), \ldots, 2^{a_2-a_1}(2^{a_1}-1)$$

(2.5.6) $$2^{a_{m-2}} - 1, 2^1(2^{a_{m-2}}-1), 2^2(2^{a_{m-2}}-1), \ldots, 2^{a_{m-1}-a_{m-2}}(2^{a_{m-2}}-1)$$

(2.5.7) $$2^{a_{m-1}} - 1, 2^h(2^{a_{m-2}}-1), 2^{h+1}(2^{a_{m-2}}-1), \ldots, 2^{a_m-a_{m-2}}(2^{a_{m-2}}-1)$$

(2.5.8) $$2^{a_m} - 1, 2^1(2^{a_m}-1), 2^2(2^{a_m}-1), \ldots, 2^{a_{m+1}-a_m}(2^{a_m}-1)$$

$$2^{a_{k-1}} - 1, 2^1(2^{a_{k-1}}-1), 2^2(2^{a_{k-1}}-1), \ldots, 2^{a_k - a_{k-1}}(2^{a_{k-1}}-1)$$

$$2^{a_k} - 1 = 2^n - 1 .$$

$h = a_{m-1} - a_{m-2} + 1$ and $a_m - a_{m-2} = (a_m - a_{m-1}) + (a_{m-1} - a_{m-2})$.

The length of this chain is $\ell^{**}(n) + n - 1$ because it contains the $\ell^{**}(n)$ numbers $2^{a_i} - 1$, $i = 1,2,\ldots,k$ and

$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_k - a_{k-1}) = a_k - 1 = n - 1$.

CHAPTER III

ON THE DIFFERENCE $\ell(n) - \ell(mn)$

§3.1. Introduction. In this chapter we examine Donald Knuth's conjecture that $\ell(n) \leq \ell(mn)$ for all m and n. As mentioned earlier (Theorem (1.5.5) Robert Giese proved that this conjecture is false for infinitely many n when $m = 3$. A year later, Edward Thurber (Theorem (1.5.7)) improved this result. We here improve this result further as follows:

(3.1.1) Theorem. For all integers $m > 1$, $m \neq 2^k$ for any $k \geq 1$ and b an arbitrary nonnegative integer, there exists infinitely many integers n for which $\ell(mn) \leq \ell(n) - b$.

We also prove that $\ell(mn) > \ell(n)$ for infinitely many n and all m. This and Theorem (3.3.1) enables us to conclude that

(3.1.2) $$\underline{\lim}_{n\to\infty} [\ell(n)-\ell(mn)] < 0 \text{ for all } m$$

and

(3.1.3) $$\overline{\lim}_{n\to\infty} [\ell(n)-\ell(mn)] = +\infty \text{ for all } m \neq 2^k.$$

At the end of the chapter we prove $\ell(n) = \ell(3n)$ and $\ell(n) = \ell(3n) - 1$ for infinitely many n.

§3.2. An Example, $\ell(5n) \leq \ell(n) - b$. Before proving Theorem (3.3.1) in general we will examine the case $m = 5$. Consider the sequence of integers $a_1, a_2, a_3, \ldots$ where $a_1 = 5 - 2$ and

$$a_i = \begin{cases} 2a_{i-1} & \text{if } 2a_{i-1} \leq 4 \\ 2a_{i-1} - 5 & \text{otherwise} \end{cases} \tag{3.2.1}$$

if $i \geq 2$. That is, we have the following sequence.

$$\begin{aligned} &1 \cdot 5 - 2^1 \\ &1 \cdot 5 - 2^2 \\ &2 \cdot 5 - 2^3 \\ &4 \cdot 5 - 2^4 \\ &7 \cdot 5 - 2^5 \\ &13 \cdot 5 - 2^6 \\ &26 \cdot 5 - 2^7 \\ &52 \cdot 5 - 2^8 \\ &103 \cdot 5 - 2^9 \\ &205 \cdot 5 - 2^{10} \\ &410 \cdot 5 - 2^{11} \\ &820 \cdot 5 - 2^{12} \\ &1639 \cdot 5 - 2^{13} \\ &3277 \cdot 5 - 2^{14} \\ &6554 \cdot 5 - 2^{15} \\ &13108 \cdot 5 - 2^{16} \end{aligned} \tag{3.2.2}$$

We now consider the subsequence obtained by choosing every fourth term of (3.2.2) beginning with $1 \cdot 5 - 2^2$. Since each term of

this subsequence equals one we get

$$1 \cdot 5 = 2^2 + 1$$

$$13 \cdot 5 = 2^6 + 1$$

(3.2.3) $$205 \cdot 5 = 2^{10} + 1$$

$$3277 \cdot 5 = 2^{14} + 1$$

$$52429 \cdot 5 = 2^{18} + 1 .$$

Note that the k^{th} term of (3.2.3) is $2^{4k-2} + 1$.

Next we consider the sequence

(3.2.4) $$1, 13, 205, 3277, 52429, \ldots$$

which can be written as

$$1 = 1$$

$$13 = 2^3 + 2^2 + 1$$

(3.2.5) $$205 = 2^7 + 2^6 + 13$$

$$3277 = 2^{11} + 2^{10} + 205$$

$$52429 = 2^{15} + 2^{14} + 3277 .$$

Note that the k^{th} term b_k of (3.2.5) is $2^{4k-5} + 2^{4k-6} + b_{k-1}$ and that $v(b_k) = 2k - 1$. Since the k^{th} term of (3.2.3) is five times the k^{th} term of (3.2.5) we have

(3.2.6) $$5(2^{4k-5}+2^{4k-6}+b_{k-1}) = 2^{4k-2} + 1$$

and

(3.2.7) $$v(2^{4k-5}+2^{4k-6}+b_{k-1}) = 2k - 1 .$$

Since $\ell(n) \geq \lambda(n) + \log_3 v(n)$ (Theorem (1.5.13)) it is easy to see that the minimal length of a chain for $2^{4k-5} + 2^{4k-6} + b_{k-1}$ will eventually become greater than b plus the minimal length of a chain for the corresponding number $2^{4k-2} + 1$.

§3.3. Lim Sup and Lim Inf of $[\ell(n)-\ell(mn)]$. First we prove Theorem (3.1.1) and then use this to show that

$$\overline{\lim_{n\to\infty}} [\ell(n)-\ell(mn)] = +\infty \quad \text{for all } m \neq 2^k . \tag{3.3.1}$$

To prove Theorem (3.1.1) we show that the construction used in section (3.2) is valid for all $m \neq 2^k$, $k = 1,2,\ldots$. Next we prove that $\ell(mn) > \ell(n)$ for infinitely many n and all m. From this we conclude that

$$\underline{\lim_{n\to\infty}} [\ell(n)-\ell(mn)] < 0 \quad \text{for all } m . \tag{3.2.2}$$

(3.3.3) Lemma. If $K > 2$ the sequence of positive integers $a_1, a_2, a_3, \ldots$ defined by $a_1 = K - 2^1$ and

$$a_p = \begin{cases} 2a_{p-1} & \text{if } 2a_{p-1} \leq K - 1 \\ 2a_{p-1} - K & \text{otherwise} \end{cases}$$

if $p \geq 2$ is cyclic.

Proof. Since $1 \leq a_i \leq K - 1$ for all i we have $a_1 = a_j$ for some j, $1 < j \leq K$. Because of the way the sequence is defined

$$a_1 = a_j \Rightarrow a_2 = a_{j+1} \Rightarrow a_3 = a_{j+2} \Rightarrow \ldots$$

(3.3.4) Lemma. If $K - 2^a = 2b + 1$, $K \geq 3$, $2^a > 2b + 1 \geq 1$ then there exists r, $r = 2c + 1 \geq 3$ such that $Kr - 2^{a+d} = 2b + 1$ for some $d \geq 1$.

Proof. Lemma (3.3.3) guarantees the existence of an $r \geq 3$ such that $Kr - 2^{a+d} = 2b + 1$ for some $d \geq 1$. That r is odd follows from the fact that $K - 2^a$ is odd.

(3.3.5) Remark. The fact that r is odd is required in the proof of the next lemma.

(3.3.6) Lemma. If $K = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_r} + 2^0$ and $R_0 = 1$ then there exists $R_u = 2^{b_1+(u-1)d} + 2^{b_2+(u-1)d} + \ldots + 2^{b_s+(u-1)d} + R_{u-1}$, $u = 1,2,3,\ldots$ such that $KR_u = 2^{a_1+ud} + 2^{a_2} + 2^{a_3} + \ldots + 2^{a_r} + 2^0$.

Proof. The proof is by induction on u. By Lemma (3.3.4) there exists

$$R_1 = 2^{b_1} + 2^{b_2} + \ldots + 2^{b_s} + R_0$$

such that

$$KR_1 = 2^{a_1+d} + 2^{a_2} + \ldots + 2^{a_r} + 2^0 .$$

Suppose there exists R_u, $u \geq 1$

$$R_u = 2^{b_1+(u-1)d} + 2^{b_2+(u-1)d} + \ldots + 2^{b_s+(u-1)d} + R_{u-1}$$

such that

$$KR_u = 2^{a_1+ud} + 2^{a_2} + 2^{a_3} + \ldots + 2^{a_r} + 2^0 . \tag{3.3.7}$$

To complete the proof we must show that there exists R_{u+1} :

$$R_{u+1} = 2^{b_1+ud} + 2^{b_2+ud} + \ldots + 2^{b_s+ud} + R_u$$

such that

$$KR_{u+1} = 2^{a_1+(u+1)d} + 2^{a_2} + 2^{a_3} + \ldots + 2^{a_r} + 2^0 .$$

Clearly

$$K(R_1-1) = KR_1 - K = 2^{a_1+d} - 2^{a_1} .$$

Therefore

$$K[2^{ud}(R_1-1)] = 2^{ud}(2^{a_1+d} - 2^{a_1}) = 2^{a_1+(u+1)d} - 2^{a_1+ud} .$$

Adding KR_u to each side of this equality we get

$$K[2^{ud}(R_1-1)+R_u] = 2^{a_1+(u+1)d} - 2^{a_1+ud} + KR_u .$$

That is,

$$KR_{u+1} = 2^{a_1+(u+1)d} - 2^{a_1+ud} + KR_u .$$

Using (3.3.7) we get

$$KR_{u+1} = 2^{a_1+(u+1)d} + 2^{a_2} + 2^{a_3} + \ldots + 2^{a_r} + 2^0 .$$

(3.3.8) Theorem. For all integers $m > 1$, $m \neq 2^h$ for any $h \geq 1$ and b an arbitrary nonnegative integer, there exists infinitely many integers n for which $\ell(mn) \leq \ell(n) - b$.

Proof. For all odd K we will show that

$$\ell(KR_u) < \ell(R_u) - b$$

if u is sufficiently large.

$$KR_u = 2^{a_1+ud} + 2^{a_2} + \ldots + 2^{a_r} + 2^0$$

implies

$$\ell(KR_u) \leq a_1 + ud + r .$$

$$R_u = 2^{b_1+(u-1)d} + 2^{b_2+(u-1)d} + \ldots + 2^{b_s+(u-1)d} + R_{u-1}$$

implies $v(R_u) \geq u + 1$. Also

$$\ell(n) \geq \lambda(n) + \log_3 v(n)$$

implies

$$\ell(R_u) \geq b_1 + ud - d + \log_3(u+1) .$$

If u is sufficiently large $a_1 + ud + r \leq b_1 + ud - d + \log_3(u+1) - b$ since a_1, d, r, b_1 and b are all fixed. That is $\ell(KR_u) \leq \ell(R_u) - b$ for all odd K.

To complete the proof we must show that given any v,

$$\ell(2^v KR_u) \leq \ell(R_u) - b$$

if u is sufficiently large.

$$\ell(2^v KR_u) \leq \ell(2^v) + \ell(KR_u) \leq v + \ell(KR_u)$$
$$\leq v + \ell(R_u) - b .$$

(3.3.9) Remark. The identity

$$\frac{2^{4rs+2r}+2^0}{2^{2r}+2^0} = 2^{4rs-1}+2^{4rs-2}+2^{4rs-3}+\ldots + 2^{4rs-2r+1}+2^{4rs-2r}+\frac{2^{4rs-2r}+2^0}{2^{2r}+2^0}$$

can be used to prove Theorem (3.3.8) when $m = 2^{2r}+2^0$, $r \geq 1$.

(3.3.10) Theorem. For each $m > 1$ there exists infinitely many n such that $\ell(mn) > \ell(n)$.

Proof. Choosing $n = 2^k$, $k \geq 1$ we get

$$\ell(mn) = \ell(m2^k) \geq k+1 > k = \ell(2^k) = \ell(n).$$

(3.3.11) Unsolved Problem. Does there exist a constant R such that $\ell(n) \leq \ell(Rn)$ for all n? Theorem (3.3.8) shows that such a constant would have to be a power of two. W.R. Utz [20] conjectured that $\ell(2n) = \ell(n)+1$. We now know this conjecture is false because Edward Thurber [19] proved $\ell(2n) = \ell(n)$ for infinitely many n. The conjecture has therefore been changed to $\ell(2n) \geq \ell(n)$.

Combining Theorems (3.3.8) and (3.3.10) we obtain.

(3.3.12) Theorem. $\underline{\lim}_{n\to\infty}\,[\ell(n)-\ell(mn)] < 0$ for all m and $\overline{\lim}_{n\to\infty}\,[\ell(n)-\ell(mn)] = +\infty$ for all $m \neq 2^k$, $k \geq 1$.

§3.4. $\underline{\ell(n) = \ell(3n),\ \ell(n) = \ell(3n) - 1}$. Generally it is not known when $\ell(mn) = \ell(n) - b$. We therefore give a few specific cases when this is true.

(3.4.1) Lemma. Case I. $\ell(2^{A+7}+2^{A+5}+2^{A+3}+2^{A+1}+2^{A}) = A + 10$.

Case II. $\ell(2^{A+9}+2^{A+7}+2^{A+5}+2^{A+3}+2^{A+1}+2^{A}) = A + 12$.

Proof. The following chain for $2^{A+7} + 2^{A+5} + 2^{A+3} + 2^{A+1} + 2^{A}$ is of length $A + 10$ (all steps indicated by "..." represent doublings).

$2^0, 2^1, \ldots, 2^A, 2^{A+1}, 2^{A+2}, 2^{A+3}, 2^{A+3}+2^{A+1}, 2^1(2^{A+3}+2^{A+1}), \ldots, 2^4(2^{A+3}+2^{A+1}) =$
$2^{A+7} + 2^{A+5},\ 2^{A+7} + 2^{A+5} + 2^{A+3} + 2^{A+1},\ 2^{A+7} + 2^{A+5} + 2^{A+3} + 2^{A+1} + 2^{A}$.

The following chain for $2^{A+9} + 2^{A+7} + 2^{A+5} + 2^{A+3} + 2^{A+1} + 2^{A}$ is of length $A + 12$.

$2^0, 2^1, \ldots, 2^A, 2^{A+1}, 2^{A+2}, 2^{A+3}, 2^{A+3}+2^{A+1}, 2^{A+3}+2^{A+1}+2^{A}, 2^{A+4}+2^{A+2}+2^{A},$
$2^1(2^{A+4}+2^{A+2}+2^{A}), \ldots, 2^5(2^{A+4}+2^{A+2}+2^{A}) = 2^{A+9} + 2^{A+7} + 2^{A+5},$
$2^{A+9} + 2^{A+7} + 2^{A+5} + 2^{A+3} + 2^{A+1} + 2^{A}$.

Since

$$\ell(2^{A_1}+2^{A_2}+\ldots+2^{A_m}) \geq A_1 + 3$$

if $m \geq 5$ the proof is complete.

(3.4.2) Theorem. (A) $\ell(2^k \cdot 11) = \ell(3 \cdot 2^k \cdot 11) - 1,\ k \geq 0$

(B) $\ell(2^k \cdot 43) = \ell(3 \cdot 2^k \cdot 43) - 1,\ k \geq 0$

(C) $\ell(2^k \cdot 171) = \ell(3 \cdot 2^k \cdot 171),\ k \geq 0$

(D) $\ell(2^k \cdot 683) = \ell(3 \cdot 2^k \cdot 683),\ k \geq 0$.

Proof. (A) $\ell(2^k \cdot 11) = \ell(2^{k+3}+2^{k+1}+2^k) = k + 5$

by Theorem (1.3.1) and

$$\ell(3 \cdot 2^k \cdot 11) = \ell(2^{k+5}+2^k) = k + 6$$

by Theorem (1.1.16).

(B) $\ell(2^k \cdot 43) = \ell(2^{k+5}+2^{k+3}+2^{k+1}+2^k) = k + 7$

by Theorem (1.4.7) and

$$\ell(3 \cdot 2^k \cdot 43) = \ell(2^{k+7}+2^k) = k + 8$$

by Theorem (1.1.16).

(C) $\ell(2^k \cdot 171) = \ell(2^{k+7}+2^{k+5}+2^{k+3}+2^{k+1}+2^k) = k + 10$

by Lemma (3.4.1) and

$$\ell(3 \cdot 2^k \cdot 171) = \ell(2^{k+9}+ 2^k) = k + 10$$

by Theorem (1.1.16).

(D) $\ell(2^k \cdot 683) = \ell(2^{k+9}+2^{k+7}+2^{k+5}+2^{k+3}+2^{k+1}+2^k) = k + 12$

by Lemma (3.4.1) and

$$\ell(3 \cdot 2^k \cdot 683) = \ell(2^{k+11}+2^k) = k + 12$$

by Theorem (1.1.16).

(3.4.3) Remark. 191 is the smallest number such that $\ell(n) = \ell(2n)$ and 171 is the smallest number such that $\ell(n) = \ell(mn)$ for any $m > 2$.

CHAPTER IV

CONJECTURES AND UNSOLVED PROBLEMS

In this final chapter we state some of the many conjectures and unsolved problems about addition chains. For completeness we include some conjectures and problems that have been mentioned earlier.

One of the first questions [12] asked about addition chains is still unsettled. This question, which is now called the Scholz-Brauer conjecture, states that

$$\ell(2^n-1) \leq n - 1 + \ell(n) \tag{4.1.1}$$

for all n. A. Brauer [2] proved that

$$\ell(2^n-1) \leq n - 1 + \ell^*(n) \tag{4.1.2}$$

for all n. Therefore, if inequality (4.1.1) is false for some n, this n would have to be greater than or equal to 12509 since this is the first n such that $\ell(n) \neq \ell^*(n)$. Inequality (4.1.2) remains true if $\ell^*(n)$ is replaced by $\ell^{**}(n)$ or $\ell^\circ(n)$. No n is known for which $\ell^{**}(n) \neq \ell(n)$ or $\ell^\circ(n) \neq \ell(n)$ but whether or not $\ell^{**}(n) = \ell(n)$ and $\ell^\circ(n) = \ell(n)$ for all n are still open questions.

Computer calculations performed by Donald Knuth [11] show that equality holds in (4.1.1) for $1 \leq n \leq 14$. Edward Thurber [17] has extended this list to include $n = 15, 16, 17, 18, 20, 24, 32$.

Since no n is known for which (4.1.1) is an inequality one may wonder if this is true for all n. Kenneth Stolarsky and others have conjectured that

(4.1.3) $$\ell(n) \geq \lambda(n) + \log_2 v(n)$$

for all n. If this conjecture is true then it would follow that (4.1.1) is an equality for infinitely many n. That is, (4.1.1) would be an equality for all n of the form $2^{m+k} + 2^m$. Inequality (4.1.3) can be written as follows:

(4.1.4) $\ell(n) \geq \lambda(n) + k + 1$ if $v(n) \geq 2^k + 1$, $k = 0,1,2,\ldots$.

This conjecture has only been proven for $k = 0,1,2,3$. Edward Thurber's [19] proof of this conjecture for $k = 3$ required a detailed examination of 120 cases.

The Scholz-Brauer conjecture implies

(4.1.5) $$\ell(2^{n+1}-2) \leq n + \ell(n)$$

since $\ell(2^{n+1}-2) = \ell((2)(2^n-1)) \leq \ell(2) + \ell(2^n-1)$. M.V. Subbarao would like to know if (4.1.5) implies the Scholz-Brauer conjecture.

Hansen proved that

(4.1.6) $$\ell^*(n) = \lambda(n) + v(n) - 1$$

for some n. Is equality (4.1.6) true if ℓ^* is replaced by ℓ^{**}, ℓ° or ℓ? If this is true for ℓ then the best upper bound one can obtain for $\ell(2^{b_0}+2^{b_1}+\ldots+2^{b_m})$ is $b_0 + m$. We know there exists n

such that $\ell(n) = \lambda(n) + v(n) - 1$ when $v(n) = 1, 2, 3, 4, 5$ and 6.

Can the upper bounds for $\ell(n)$ be improved? For example, do the following hold for all n ?

(4.1.6) $\ell(n) \leq (r+1)s + 2r - 2$ for $2^{rs} \leq n < 2^{r(s+1)}$ $r \geq 1$, $s \geq 0$

(4.1.7) $\ell(n) \leq \lambda(n) + (k-1) + [\lambda(n)/k]$ for any $k \geq 1$.

Inequalities (4.1.6) and (4.1.7) are true for all $n < 18{,}270$. These inequalities give the same bounds when $|r-s|$ is small (i.e. $r - s = \pm 1$, $r = s$) but in other cases (4.1.7) gives the better bound.

Let

(4.1.8) $$h(m) = \max_{n \leq 2^m} \ell(n) .$$

If

(4.1.9) $$h(m-1) \leq h(m) - 1$$

for all $m \geq 1$ then inequality (4.1.6) would imply inequality (4.1.7). Using (4.1.2) it is easily shown that (4.1.6) is true for $n = 2^{r(s+1)} - 1$, the n with the greatest number of ones in its binary representation between 2^{rs} and $2^{r(s+1)}$. This suggests an investigation of the difference

(4.1.10) $$h(m) - \ell(2^m-1)$$

(4.1.10) equals one when $m = 8$ and 12 and it equals zero in all other cases when its value is known. Is

(4.1.11) $$\overline{\lim_{m\to\infty}}\,(h(m)-\ell(2^m-1)) < +\infty$$

and is

(4.1.12) $$\lim_{m\to\infty}\,(h(m)-\ell(2^m-1)) = 0\ ?$$

A.A. Gioia, M.V. Subbarao, and M. Sugunamma [6] proved that the Scholz-Brauer conjecture was true for $2n$ if it was true for n and $\ell(2n) = \ell(n) + 1$. Therefore we would like to know when

(4.1.13) $$\ell(2n) = \ell(n) + 1\ .$$

At first it was conjectured that (4.1.13) held for all n, but computer calculations produced several counterexamples. Then Edward Thurber [19] proved that (4.1.13) was false for infinitely many n by showing

(4.1.14) $\ell(2n) = \ell(n) = m + 8$ if $n = 2^m(23) + 7$ and $m \geq 5$.

It is easy to check that $\ell(23) = 6$; in fact, there are only four chains of length 6 for the number 23. They are: 1, 2, 3, 5, 10, 13, 23; 1, 2, 3, 5, 10, 20, 23; 1, 2, 4, 5, 9, 14, 23; 1, 2, 4, 5, 9, 18, 23. Notice that the number 14 appears in one of these chain for 23 but the number 7 does not appear in any of them. The chains of minimal length for 35 also have this property; that is, the number 22 can appear in a chain of minimal length for 35 but 11 cannot. Does

(4.1.15) $\ell(2n) = \ell(n)$ when $n = 2^m(35) + 11$ and $m \geq 7$?

This is true when $m = 7$ and 8 but beyond this nothing is known. In general, if $2k$ appears in a chain of minimal length for p and k

does not (not necessarily in the same minimal chain), does this imply $\ell(2n) = \ell(n)$ when $n = 2^m(p) + k$ and m is sufficiently large? Are these the only times when $\ell(2n) = \ell(n)$? Does there exist a constant k independent of n such that

(4.1.16) $$\ell(n) \leq \ell(2^k n)$$

for all n, in particular is

(4.1.17) $$\ell(n) \leq \ell(2n)$$

for all n ? Does there exist infinitely many n so that

(4.1.18) $$\ell(n) = \ell(2^k n)$$

for any $k > 1$?

Very little is known about the functions $c(r)$ and $d(r)$. We proved that

(4.1.19) $$\log_2 c(r) \sim r$$

but were unable to prove that

(4.1.20) $$\log_2 d(r) \sim r$$

without the assumption that $d(r)$ was monotonic increasing. Proving that $d(r)$ is monotonic increasing, if in fact it is, appears to be a very difficult question. If one could find all n such that

(4.1.21) $$\ell(n) = \lambda(n) + k, \quad k = 0,1,2,3,\ldots$$

we could then determine $d_k(r)$ and thus obtain $d(r)$. Some n are

known for which $\ell(n) = \lambda(n) + k$, $k = 3,4$; but all solutions of (4.1.21) are known only when $k = 0,1$ and 2 . What are the asymptotic values of $c(r)$ and $d(r)$? Investigate

$$\frac{1}{x} \sum_{r \le x} c(r) \tag{4.1.22}$$

and

$$\frac{1}{x} \sum_{r \le x} d(r) . \tag{4.1.23}$$

Are the ratios $c(r+1)/c(r)$ and $d(r+1)/d(r)$ less than or equal to some constant? Investigate $d^*(r)$, where $d^*(r)$ is the number of solutions of the equation $\ell^*(n) = r$. Is $d^*(r)$ monotonic increasing?

We have seen that $x\ell(x) = y\ell(y)$ and $\ell(x)/x = \ell(y)/y$ both have an infinite number of solutions (x,y) . Does

$$x\ell(x) = y\ell(x) = z\ell(z) \tag{4.1.24}$$

and

$$\ell(x)/x = \ell(y)/y = \ell(z)/z \tag{4.1.25}$$

each have an infinite number of solutions (x,y,z) ? What happens if we have n variables? Investigate the distribution of values of $\ell(n)/n$; for example, what is the minimal and maximal value of $\ell(n)/n$ if $2^m \le n < 2^{m+1}$? What is the average value of $\ell(n)$, i.e. $\frac{1}{x} \sum_{n \le x} \ell(n)$? Explore similar situations when ℓ is replaced by ℓ^* or ℓ° .

One could also ask most of these questions about addition-subtraction chains. That is, a chain where the condition $a_i = a_j + a_k$ $(0 \le k \le j \le i-1)$ is replaced by $a_i = a_j \pm a_k$ $(0 \le k \le j \le i-1)$.

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APPENDIX I

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
1	0	0.0	51	7	0.1373	101	9	0.0891
2	1	0.5000	52	7	0.1346	102	8	0.0784
3	2	0.6667	53	8	0.1509	103	9	0.0874
4	2	0.5000	54	7	0.1296	104	8	0.0769
5	3	0.6000	55	8	0.1455	105	9	0.0857
6	3	0.5000	56	7	0.1250	106	9	0.0849
7	4	0.5714	57	8	0.1404	107	9	0.0841
8	3	0.3750	58	8	0.1379	108	8	0.0741
9	4	0.4444	59	8	0.1356	109	9	0.0826
10	4	0.4000	60	7	0.1167	110	9	0.0818
11	5	0.4545	61	8	0.1311	111	9	0.0811
12	4	0.3333	62	8	0.1290	112	8	0.0714
13	5	0.3846	63	8	0.1270	113	9	0.0796
14	5	0.3571	64	6	0.0938	114	9	0.0789
15	5	0.3333	65	7	0.1077	115	9	0.0783
16	4	0.2500	66	7	0.1061	116	9	0.0776
17	5	0.2941	67	8	0.1194	117	9	0.0769
18	5	0.2778	68	7	0.1029	118	9	0.0763
19	6	0.3158	69	8	0.1159	119	9	0.0756
20	5	0.2500	70	8	0.1143	120	8	0.0667
21	6	0.2857	71	9	0.1268	121	9	0.0744
22	6	0.2727	72	7	0.0972	122	9	0.0738
23	6	0.2609	73	8	0.1096	123	9	0.0732
24	5	0.2083	74	8	0.1081	124	9	0.0726
25	6	0.2400	75	8	0.1067	125	9	0.0720
26	6	0.2308	76	8	0.1053	126	9	0.0714
27	6	0.2222	77	8	0.1039	127	10	0.0787
28	6	0.2143	78	8	0.1026	128	7	0.0547
29	7	0.2414	79	9	0.1139	129	8	0.0620
30	6	0.2000	80	7	0.0875	130	8	0.0615
31	7	0.2258	81	8	0.0988	131	9	0.0687
32	5	0.1563	82	8	0.0976	132	8	0.0606
33	6	0.1818	83	8	0.0964	133	9	0.0677
34	6	0.1765	84	8	0.0952	134	9	0.0672
35	7	0.2000	85	8	0.0941	135	9	0.0667
36	6	0.1667	86	8	0.0930	136	8	0.0588
37	7	0.1892	87	9	0.1034	137	9	0.0657
38	7	0.1842	88	8	0.0909	138	9	0.0652
39	7	0.1795	89	9	0.1011	139	10	0.0719
40	6	0.1500	90	8	0.0889	140	9	0.0643
41	7	0.1707	91	9	0.0989	141	10	0.0709
42	7	0.1667	92	8	0.0870	142	10	0.0704
43	7	0.1628	93	9	0.0968	143	10	0.0699
44	7	0.1591	94	9	0.0957	144	8	0.0556
45	7	0.1556	95	9	0.0947	145	9	0.0621
46	7	0.1522	96	7	0.0729	146	9	0.0616
47	8	0.1702	97	8	0.0825	147	9	0.0612
48	6	0.1250	98	8	0.0816	148	9	0.0608
49	7	0.1429	99	8	0.0808	149	9	0.0604
50	7	0.1400	100	8	0.0800	150	9	0.0600

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
151	10	0.0662	201	10	0.0498	251	11	0.0438
152	9	0.0592	202	10	0.0495	252	10	0.0397
153	9	0.0588	203	10	0.0493	253	11	0.0435
154	9	0.0584	204	9	0.0441	254	11	0.0433
155	10	0.0645	205	10	0.0488	255	10	0.0392
156	9	0.0577	206	10	0.0485	256	8	0.0312
157	10	0.0637	207	10	0.0483	257	9	0.0350
158	10	0.0633	208	9	0.0433	258	9	0.0349
159	10	0.0629	209	10	0.0478	259	10	0.0386
160	8	0.0500	210	10	0.0476	260	9	0.0346
161	9	0.0559	211	10	0.0474	261	10	0.0383
162	9	0.0556	212	10	0.0472	262	10	0.0382
163	9	0.0552	213	10	0.0469	263	11	0.0418
164	9	0.0549	214	10	0.0467	264	9	0.0341
165	9	0.0545	215	10	0.0465	265	10	0.0377
166	9	0.0542	216	9	0.0417	266	10	0.0376
167	10	0.0599	217	10	0.0461	267	11	0.0412
168	9	0.0536	218	10	0.0459	268	10	0.0373
169	10	0.0592	219	10	0.0457	269	11	0.0409
170	9	0.0529	220	10	0.0455	270	10	0.0370
171	10	0.0585	221	10	0.0452	271	11	0.0406
172	9	0.0523	222	10	0.0450	272	9	0.0331
173	10	0.0578	223	11	0.0493	273	10	0.0366
174	10	0.0575	224	9	0.0402	274	10	0.0365
175	10	0.0571	225	10	0.0444	275	11	0.0400
176	9	0.0511	226	10	0.0442	276	10	0.0362
177	10	0.0565	227	10	0.0441	277	11	0.0397
178	10	0.0562	228	10	0.0439	278	11	0.0396
179	10	0.0559	229	10	0.0437	279	11	0.0394
180	9	0.0500	230	10	0.0435	280	10	0.0357
181	10	0.0552	231	10	0.0433	281	10	0.0356
182	10	0.0549	232	10	0.0431	282	11	0.0390
183	10	0.0546	233	10	0.0429	283	11	0.0389
184	9	0.0489	234	10	0.0427	284	11	0.0387
185	10	0.0541	235	11	0.0468	285	11	0.0386
186	10	0.0538	236	10	0.0424	286	11	0.0385
187	10	0.0535	237	11	0.0464	287	11	0.0383
188	10	0.0532	238	10	0.0420	288	9	0.0312
189	10	0.0529	239	11	0.0460	289	10	0.0346
190	10	0.0526	240	9	0.0375	290	10	0.0345
191	11	0.0576	241	10	0.0415	291	10	0.0344
192	8	0.0417	242	10	0.0413	292	10	0.0342
193	9	0.0466	243	10	0.0412	293	10	0.0341
194	9	0.0464	244	10	0.0410	294	10	0.0340
195	9	0.0462	245	10	0.0408	295	11	0.0373
196	9	0.0459	246	10	0.0407	296	10	0.0338
197	10	0.0508	247	11	0.0445	297	10	0.0337
198	9	0.0455	248	10	0.0403	298	10	0.0336
199	10	0.0503	249	10	0.0402	299	11	0.0368
200	9	0.0450	250	10	0.0400	300	10	0.0333

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
301	11	0.0365	351	11	0.0313	401	11	0.0274
302	11	0.0364	352	10	0.0284	402	11	0.0274
303	11	0.0363	353	11	0.0312	403	11	0.0273
304	10	0.0329	354	11	0.0311	404	11	0.0272
305	11	0.0361	355	11	0.0310	405	11	0.0272
306	10	0.0327	356	11	0.0309	406	11	0.0271
307	11	0.0358	357	11	0.0308	407	12	0.0295
308	10	0.0325	358	11	0.0307	408	10	0.0245
309	11	0.0356	359	11	0.0306	409	11	0.0269
310	11	0.0355	360	10	0.0278	410	11	0.0268
311	11	0.0354	361	11	0.0305	411	11	0.0268
312	10	0.0321	362	11	0.0304	412	11	0.0267
313	11	0.0351	363	11	0.0303	413	11	0.0266
314	11	0.0350	364	11	0.0302	414	11	0.0266
315	11	0.0349	365	11	0.0301	415	11	0.0265
316	11	0.0348	366	11	0.0301	416	10	0.0240
317	11	0.0347	367	11	0.0300	417	11	0.0264
318	11	0.0346	368	10	0.0272	418	11	0.0263
319	11	0.0345	369	11	0.0298	419	11	0.0263
320	9	0.0281	370	11	0.0297	420	11	0.0262
321	10	0.0312	371	11	0.0296	421	11	0.0261
322	10	0.0311	372	11	0.0296	422	11	0.0261
323	10	0.0310	373	11	0.0295	423	11	0.0260
324	10	0.0309	374	11	0.0294	424	11	0.0259
325	10	0.0308	375	11	0.0293	425	11	0.0259
326	10	0.0307	376	11	0.0293	426	11	0.0258
327	11	0.0336	377	11	0.0292	427	12	0.0281
328	10	0.0305	378	11	0.0291	428	11	0.0257
329	11	0.0334	379	12	0.0317	429	11	0.0256
330	10	0.0303	380	11	0.0289	430	11	0.0256
331	11	0.0332	381	11	0.0289	431	12	0.0278
332	10	0.0301	382	11	0.0288	432	10	0.0231
333	11	0.0330	383	12	0.0313	433	11	0.0254
334	11	0.0329	384	9	0.0234	434	11	0.0253
335	11	0.0328	385	10	0.0260	435	11	0.0253
336	10	0.0298	386	10	0.0259	436	11	0.0252
337	11	0.0326	387	10	0.0258	437	11	0.0252
338	11	0.0325	388	10	0.0258	438	11	0.0251
339	11	0.0324	389	11	0.0283	439	12	0.0273
340	10	0.0294	390	10	0.0256	440	11	0.0250
341	11	0.0323	391	11	0.0281	441	11	0.0249
342	11	0.0322	392	10	0.0255	442	11	0.0249
343	11	0.0321	393	11	0.0280	443	12	0.0271
344	10	0.0291	394	11	0.0279	444	11	0.0248
345	11	0.0319	395	11	0.0278	445	12	0.0270
346	11	0.0318	396	10	0.0253	446	12	0.0269
347	11	0.0317	397	11	0.0277	447	11	0.0246
348	11	0.0316	398	11	0.0276	448	10	0.0223
349	11	0.0315	399	11	0.0276	449	11	0.0245
350	11	0.0314	400	10	0.0250	450	11	0.0244

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
451	11	0.0244	501	12	0.0240	551	12	0.0218
452	11	0.0243	502	12	0.0239	552	11	0.0199
453	11	0.0243	503	12	0.0239	553	11	0.0199
454	11	0.0242	504	11	0.0218	554	12	0.0217
455	11	0.0242	505	12	0.0238	555	12	0.0216
456	11	0.0241	506	12	0.0237	556	12	0.0216
457	11	0.0241	507	12	0.0237	557	12	0.0215
458	11	0.0240	508	12	0.0236	558	12	0.0215
459	11	0.0240	509	12	0.0236	559	12	0.0215
460	11	0.0239	510	11	0.0216	560	11	0.0196
461	12	0.0260	511	12	0.0235	561	11	0.0196
462	11	0.0238	512	9	0.0176	562	11	0.0196
463	12	0.0259	513	10	0.0195	563	12	0.0213
464	11	0.0237	514	10	0.0195	564	12	0.0213
465	12	0.0258	515	11	0.0214	565	12	0.0212
466	11	0.0236	516	10	0.0194	566	12	0.0212
467	12	0.0257	517	11	0.0213	567	12	0.0212
468	11	0.0235	518	11	0.0212	568	12	0.0211
469	12	0.0256	519	12	0.0231	569	12	0.0211
470	12	0.0255	520	10	0.0192	570	12	0.0211
471	12	0.0255	521	11	0.0211	571	12	0.0210
472	11	0.0233	522	11	0.0211	572	12	0.0210
473	12	0.0254	523	12	0.0229	573	12	0.0209
474	12	0.0253	524	11	0.0210	574	12	0.0209
475	12	0.0253	525	12	0.0229	575	12	0.0209
476	11	0.0231	526	12	0.0228	576	10	0.0174
477	12	0.0252	527	12	0.0228	577	11	0.0191
478	12	0.0251	528	10	0.0189	578	11	0.0190
479	12	0.0251	529	11	0.0208	579	11	0.0190
480	10	0.0208	530	11	0.0208	580	11	0.0190
481	11	0.0229	531	12	0.0226	581	11	0.0189
482	11	0.0228	532	11	0.0207	582	11	0.0189
483	11	0.0228	533	12	0.0225	583	12	0.0206
484	11	0.0227	534	12	0.0225	584	11	0.0188
485	11	0.0227	535	12	0.0224	585	11	0.0188
486	11	0.0226	536	11	0.0205	586	11	0.0188
487	12	0.0246	537	12	0.0223	587	12	0.0204
488	11	0.0225	538	12	0.0223	588	11	0.0187
489	11	0.0225	539	12	0.0223	589	12	0.0204
490	11	0.0224	540	11	0.0204	590	12	0.0203
491	12	0.0244	541	12	0.0222	591	12	0.0203
492	11	0.0224	542	12	0.0221	592	11	0.0186
493	12	0.0243	543	12	0.0221	593	12	0.0202
494	12	0.0243	544	10	0.0184	594	11	0.0185
495	11	0.0222	545	11	0.0202	595	12	0.0202
496	11	0.0222	546	11	0.0201	596	11	0.0185
497	12	0.0241	547	12	0.0219	597	12	0.0201
498	11	0.0221	548	11	0.0201	598	12	0.0201
499	12	0.0240	549	12	0.0219	599	12	0.0200
500	11	0.0220	550	12	0.0218	600	11	0.0183

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
601	12	0.0200	651	12	0.0184	701	13	0.0185
602	12	0.0199	652	11	0.0169	702	12	0.0171
603	12	0.0199	653	12	0.0184	703	13	0.0185
604	12	0.0199	654	12	0.0183	704	11	0.0156
605	12	0.0198	655	12	0.0183	705	12	0.0170
606	12	0.0198	656	11	0.0168	706	12	0.0170
607	13	0.0214	657	12	0.0183	707	12	0.0170
608	11	0.0181	658	12	0.0182	708	12	0.0169
609	12	0.0197	659	12	0.0182	709	12	0.0169
610	12	0.0197	660	11	0.0167	710	12	0.0169
611	12	0.0196	661	12	0.0182	711	12	0.0169
612	11	0.0180	662	12	0.0181	712	12	0.0169
613	12	0.0196	663	12	0.0181	713	12	0.0168
614	12	0.0195	664	11	0.0166	714	12	0.0168
615	12	0.0195	665	12	0.0180	715	12	0.0168
616	11	0.0179	666	12	0.0180	716	12	0.0168
617	12	0.0194	667	12	0.0180	717	12	0.0167
618	12	0.0194	668	12	0.0180	718	12	0.0167
619	12	0.0194	669	12	0.0179	719	13	0.0181
620	12	0.0194	670	12	0.0179	720	11	0.0153
621	12	0.0193	671	13	0.0194	721	12	0.0166
622	12	0.0193	672	11	0.0164	722	12	0.0166
623	12	0.0193	673	12	0.0178	723	12	0.0166
624	11	0.0176	674	12	0.0178	724	12	0.0166
625	12	0.0192	675	12	0.0178	725	12	0.0166
626	12	0.0192	676	12	0.0178	726	12	0.0165
627	12	0.0191	677	12	0.0177	727	13	0.0179
628	12	0.0191	678	12	0.0177	728	12	0.0165
629	12	0.0191	679	12	0.0177	729	12	0.0165
630	12	0.0190	680	11	0.0162	730	12	0.0164
631	12	0.0190	681	12	0.0176	731	12	0.0164
632	12	0.0190	682	12	0.0176	732	12	0.0164
633	12	0.0190	683	12	0.0176	733	13	0.0177
634	12	0.0189	684	12	0.0175	734	12	0.0163
635	13	0.0205	685	12	0.0175	735	12	0.0163
636	12	0.0189	686	12	0.0175	736	11	0.0149
637	12	0.0188	687	12	0.0175	737	12	0.0163
638	12	0.0188	688	11	0.0160	738	12	0.0163
639	12	0.0188	689	12	0.0174	739	12	0.0162
640	10	0.0156	690	12	0.0174	740	12	0.0162
641	11	0.0172	691	12	0.0174	741	12	0.0162
642	11	0.0171	692	12	0.0173	742	12	0.0162
643	11	0.0171	693	12	0.0173	743	13	0.0175
644	11	0.0171	694	12	0.0173	744	12	0.0161
645	11	0.0171	695	13	0.0187	745	12	0.0161
646	11	0.0170	696	12	0.0172	746	12	0.0161
647	12	0.0185	697	12	0.0172	747	12	0.0161
648	11	0.0170	698	12	0.0172	748	12	0.0160
649	12	0.0185	699	12	0.0172	749	12	0.0160
650	11	0.0169	700	12	0.0171	750	12	0.0160

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
751	13	0.0173	801	12	0.0150	851	13	0.0153
752	12	0.0160	802	12	0.0150	852	12	0.0141
753	13	0.0173	803	12	0.0149	853	13	0.0152
754	12	0.0159	804	12	0.0149	854	13	0.0152
755	13	0.0172	805	12	0.0149	855	13	0.0152
756	12	0.0159	806	12	0.0149	856	12	0.0140
757	13	0.0172	807	13	0.0161	857	13	0.0152
758	13	0.0172	808	12	0.0149	858	12	0.0140
759	12	0.0158	809	12	0.0148	859	13	0.0151
760	12	0.0158	810	12	0.0148	860	12	0.0140
761	13	0.0171	811	13	0.0160	861	13	0.0151
762	12	0.0157	812	12	0.0148	862	13	0.0151
763	13	0.0170	813	12	0.0148	863	13	0.0151
764	12	0.0157	814	13	0.0160	864	11	0.0127
765	12	0.0157	815	12	0.0147	865	12	0.0139
766	13	0.0170	816	11	0.0135	866	12	0.0139
767	13	0.0169	817	12	0.0147	867	12	0.0138
768	10	0.0130	818	12	0.0147	868	12	0.0138
769	11	0.0143	819	12	0.0147	869	12	0.0138
770	11	0.0143	820	12	0.0146	870	12	0.0138
771	11	0.0143	821	13	0.0158	871	13	0.0149
772	11	0.0142	822	12	0.0146	872	12	0.0138
773	12	0.0155	823	13	0.0158	873	12	0.0137
774	11	0.0142	824	12	0.0146	874	12	0.0137
775	12	0.0155	825	12	0.0145	875	13	0.0149
776	11	0.0142	826	12	0.0145	876	12	0.0137
777	12	0.0154	827	13	0.0157	877	13	0.0148
778	12	0.0154	828	12	0.0145	878	13	0.0148
779	12	0.0154	829	13	0.0157	879	12	0.0137
780	11	0.0141	830	12	0.0145	880	12	0.0136
781	12	0.0154	831	13	0.0156	881	13	0.0148
782	12	0.0153	832	11	0.0132	882	12	0.0136
783	12	0.0153	833	12	0.0144	883	13	0.0147
784	11	0.0140	834	12	0.0144	884	12	0.0136
785	12	0.0153	835	12	0.0144	885	13	0.0147
786	12	0.0153	836	12	0.0144	886	13	0.0147
787	12	0.0152	837	12	0.0143	887	13	0.0147
788	12	0.0152	838	12	0.0143	888	12	0.0135
789	13	0.0165	839	12	0.0143	889	13	0.0146
790	12	0.0152	840	12	0.0143	890	13	0.0146
791	13	0.0164	841	12	0.0143	891	12	0.0135
792	11	0.0139	842	12	0.0143	892	13	0.0146
793	12	0.0151	843	12	0.0142	893	13	0.0146
794	12	0.0151	844	12	0.0142	894	12	0.0134
795	12	0.0151	845	12	0.0142	895	13	0.0145
796	12	0.0151	846	12	0.0142	896	11	0.0123
797	13	0.0163	847	13	0.0153	897	12	0.0134
798	12	0.0150	848	12	0.0142	898	12	0.0134
799	13	0.0163	849	12	0.0141	899	12	0.0133
800	11	0.0137	850	12	0.0141	900	12	0.0133

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
901	12	0.0133	951	13	0.0137	1001	13	0.0130
902	12	0.0133	952	12	0.0126	1002	13	0.0130
903	12	0.0133	953	13	0.0136	1003	13	0.0130
904	12	0.0133	954	13	0.0136	1004	13	0.0129
905	12	0.0133	955	13	0.0136	1005	13	0.0129
906	12	0.0132	956	13	0.0136	1006	13	0.0129
907	13	0.0143	957	13	0.0136	1007	13	0.0129
908	12	0.0132	958	13	0.0136	1008	12	0.0119
909	13	0.0143	959	13	0.0136	1009	13	0.0129
910	12	0.0132	960	11	0.0115	1010	13	0.0129
911	13	0.0143	961	12	0.0125	1011	13	0.0129
912	12	0.0132	962	12	0.0125	1012	13	0.0128
913	13	0.0142	963	12	0.0125	1013	13	0.0128
914	12	0.0131	964	12	0.0124	1014	13	0.0128
915	13	0.0142	965	12	0.0124	1015	13	0.0128
916	12	0.0131	966	12	0.0124	1016	13	0.0128
917	13	0.0142	967	13	0.0134	1017	13	0.0128
918	12	0.0131	968	12	0.0124	1018	13	0.0128
919	13	0.0141	969	12	0.0124	1019	13	0.0128
920	12	0.0130	970	12	0.0124	1020	12	0.0118
921	13	0.0141	971	13	0.0134	1021	13	0.0127
922	13	0.0141	972	12	0.0123	1022	13	0.0127
923	13	0.0141	973	13	0.0134	1023	13	0.0127
924	12	0.0130	974	13	0.0133	1024	10	0.0098
925	13	0.0141	975	12	0.0123	1025	11	0.0107
926	13	0.0140	976	12	0.0123	1026	11	0.0107
927	13	0.0140	977	13	0.0133	1027	12	0.0117
928	12	0.0129	978	12	0.0123	1028	11	0.0107
929	13	0.0140	979	13	0.0133	1029	12	0.0117
930	13	0.0140	980	12	0.0122	1030	12	0.0117
931	13	0.0140	981	13	0.0133	1031	13	0.0126
932	12	0.0129	982	13	0.0132	1032	11	0.0107
933	13	0.0139	983	13	0.0132	1033	12	0.0116
934	13	0.0139	984	12	0.0122	1034	12	0.0116
935	13	0.0139	985	13	0.0132	1035	13	0.0126
936	12	0.0128	986	13	0.0132	1036	12	0.0116
937	13	0.0139	987	13	0.0132	1037	13	0.0125
938	13	0.0139	988	13	0.0132	1038	13	0.0125
939	13	0.0138	989	13	0.0131	1039	13	0.0125
940	13	0.0138	990	12	0.0121	1040	11	0.0106
941	13	0.0138	991	13	0.0131	1041	12	0.0115
942	13	0.0138	992	12	0.0121	1042	12	0.0115
943	13	0.0138	993	13	0.0131	1043	13	0.0125
944	12	0.0127	994	13	0.0131	1044	12	0.0115
945	13	0.0138	995	13	0.0131	1045	13	0.0124
946	13	0.0137	996	12	0.0120	1046	13	0.0124
947	13	0.0137	997	13	0.0130	1047	13	0.0124
948	13	0.0137	998	13	0.0130	1048	12	0.0115
949	13	0.0137	999	13	0.0130	1049	13	0.0124
950	13	0.0137	1000	12	0.0120	1050	13	0.0124

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
1051	13	0.0124	1101	13	0.0118	1151	14	0.0122
1052	13	0.0124	1102	13	0.0118	1152	11	0.0095
1053	13	0.0123	1103	13	0.0118	1153	12	0.0104
1054	13	0.0123	1104	12	0.0109	1154	12	0.0104
1055	13	0.0123	1105	12	0.0109	1155	12	0.0104
1056	11	0.0104	1106	12	0.0108	1156	12	0.0104
1057	12	0.0114	1107	13	0.0117	1157	12	0.0104
1058	12	0.0113	1108	13	0.0117	1158	12	0.0104
1059	13	0.0123	1109	13	0.0117	1159	13	0.0112
1060	12	0.0113	1110	13	0.0117	1160	12	0.0103
1061	13	0.0123	1111	14	0.0126	1161	12	0.0103
1062	13	0.0122	1112	13	0.0117	1162	12	0.0103
1063	13	0.0122	1113	13	0.0117	1163	13	0.0112
1064	12	0.0113	1114	13	0.0117	1164	12	0.0103
1065	13	0.0122	1115	13	0.0117	1165	13	0.0112
1066	13	0.0122	1116	13	0.0116	1166	13	0.0111
1067	13	0.0122	1117	13	0.0116	1167	13	0.0111
1068	13	0.0122	1118	13	0.0116	1168	12	0.0103
1069	13	0.0122	1119	13	0.0116	1169	13	0.0111
1070	13	0.0121	1120	12	0.0107	1170	12	0.0103
1071	13	0.0121	1121	13	0.0116	1171	13	0.0111
1072	12	0.0112	1122	12	0.0107	1172	12	0.0102
1073	12	0.0112	1123	13	0.0116	1173	13	0.0111
1074	13	0.0121	1124	12	0.0107	1174	13	0.0111
1075	13	0.0121	1125	13	0.0116	1175	13	0.0111
1076	13	0.0121	1126	13	0.0115	1176	12	0.0102
1077	13	0.0121	1127	13	0.0115	1177	13	0.0110
1078	13	0.0121	1128	13	0.0115	1178	13	0.0110
1079	13	0.0120	1129	13	0.0115	1179	13	0.0110
1080	12	0.0111	1130	13	0.0115	1180	13	0.0110
1081	13	0.0120	1131	13	0.0115	1181	13	0.0110
1082	13	0.0120	1132	13	0.0115	1182	13	0.0110
1083	13	0.0120	1133	13	0.0115	1183	14	0.0118
1084	13	0.0120	1134	13	0.0115	1184	12	0.0101
1085	13	0.0120	1135	13	0.0115	1185	13	0.0110
1086	13	0.0120	1136	13	0.0114	1186	13	0.0110
1087	14	0.0129	1137	13	0.0114	1187	13	0.0110
1088	11	0.0101	1138	13	0.0114	1188	12	0.0101
1089	12	0.0110	1139	13	0.0114	1189	13	0.0109
1090	12	0.0110	1140	13	0.0114	1190	13	0.0109
1091	13	0.0119	1141	13	0.0114	1191	13	0.0109
1092	12	0.0110	1142	13	0.0114	1192	12	0.0101
1093	13	0.0119	1143	13	0.0114	1193	13	0.0109
1094	13	0.0119	1144	13	0.0114	1194	13	0.0109
1095	13	0.0119	1145	13	0.0114	1195	14	0.0117
1096	12	0.0109	1146	13	0.0113	1196	13	0.0109
1097	12	0.0109	1147	13	0.0113	1197	13	0.0109
1098	13	0.0118	1148	13	0.0113	1198	13	0.0109
1099	13	0.0118	1149	14	0.0122	1199	14	0.0117
1100	13	0.0118	1150	13	0.0113	1200	12	0.0100

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
1201	13	0.0108	1251	13	0.0104	1301	13	0.0100
1202	13	0.0108	1252	13	0.0104	1302	13	0.0100
1203	13	0.0108	1253	13	0.0104	1303	13	0.0100
1204	13	0.0108	1254	13	0.0104	1304	12	0.0092
1205	13	0.0108	1255	13	0.0104	1305	13	0.0100
1206	13	0.0108	1256	13	0.0104	1306	13	0.0100
1207	13	0.0108	1257	13	0.0103	1307	13	0.0099
1208	13	0.0108	1258	13	0.0103	1308	13	0.0099
1209	13	0.0108	1259	14	0.0111	1309	13	0.0099
1210	13	0.0107	1260	13	0.0103	1310	13	0.0099
1211	14	0.0116	1261	13	0.0103	1311	13	0.0099
1212	13	0.0107	1262	13	0.0103	1312	12	0.0091
1213	13	0.0107	1263	13	0.0103	1313	13	0.0099
1214	14	0.0115	1264	13	0.0103	1314	13	0.0099
1215	13	0.0107	1265	13	0.0103	1315	13	0.0099
1216	12	0.0099	1266	13	0.0103	1316	13	0.0099
1217	13	0.0107	1267	14	0.0110	1317	13	0.0099
1218	13	0.0107	1268	13	0.0103	1318	13	0.0099
1219	13	0.0107	1269	13	0.0102	1319	14	0.0106
1220	13	0.0107	1270	14	0.0110	1320	12	0.0091
1221	13	0.0106	1271	14	0.0110	1321	13	0.0098
1222	13	0.0106	1272	13	0.0102	1322	13	0.0098
1223	13	0.0106	1273	13	0.0102	1323	13	0.0098
1224	12	0.0098	1274	13	0.0102	1324	13	0.0098
1225	13	0.0106	1275	13	0.0102	1325	13	0.0098
1226	13	0.0106	1276	13	0.0102	1326	13	0.0098
1227	13	0.0106	1277	14	0.0110	1327	14	0.0106
1228	13	0.0106	1278	13	0.0102	1328	12	0.0090
1229	13	0.0106	1279	14	0.0109	1329	13	0.0098
1230	13	0.0106	1280	11	0.0086	1330	13	0.0098
1231	13	0.0106	1281	12	0.0094	1331	13	0.0098
1232	12	0.0097	1282	12	0.0094	1332	13	0.0098
1233	13	0.0105	1283	12	0.0094	1333	13	0.0098
1234	13	0.0105	1284	12	0.0093	1334	13	0.0097
1235	13	0.0105	1285	12	0.0093	1335	13	0.0097
1236	13	0.0105	1286	12	0.0093	1336	13	0.0097
1237	13	0.0105	1287	13	0.0101	1337	13	0.0097
1238	13	0.0105	1288	12	0.0093	1338	13	0.0097
1239	13	0.0105	1289	13	0.0101	1339	14	0.0105
1240	13	0.0105	1290	12	0.0093	1340	13	0.0097
1241	13	0.0105	1291	13	0.0101	1341	13	0.0097
1242	13	0.0105	1292	12	0.0093	1342	14	0.0104
1243	14	0.0113	1293	13	0.0101	1343	14	0.0104
1244	13	0.0105	1294	13	0.0100	1344	12	0.0089
1245	13	0.0104	1295	13	0.0100	1345	13	0.0097
1246	13	0.0104	1296	12	0.0093	1346	13	0.0097
1247	14	0.0112	1297	13	0.0100	1347	13	0.0097
1248	12	0.0096	1298	13	0.0100	1348	13	0.0096
1249	13	0.0104	1299	13	0.0100	1349	13	0.0096
1250	13	0.0104	1300	12	0.0092	1350	13	0.0096

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
1351	13	0.0096	1401	14	0.0100	1451	14	0.0096
1352	13	0.0096	1402	13	0.0093	1452	13	0.0090
1353	13	0.0096	1403	14	0.0100	1453	14	0.0096
1354	13	0.0096	1404	13	0.0093	1454	14	0.0096
1355	13	0.0096	1405	13	0.0093	1455	13	0.0089
1356	13	0.0096	1406	14	0.0100	1456	13	0.0089
1357	13	0.0096	1407	14	0.0100	1457	14	0.0096
1358	13	0.0096	1408	12	0.0085	1458	13	0.0089
1359	13	0.0096	1409	13	0.0092	1459	14	0.0096
1360	12	0.0088	1410	13	0.0092	1460	13	0.0089
1361	13	0.0096	1411	13	0.0092	1461	14	0.0096
1362	13	0.0095	1412	13	0.0092	1462	13	0.0089
1363	13	0.0095	1413	13	0.0092	1463	14	0.0096
1364	13	0.0095	1414	13	0.0092	1464	13	0.0089
1365	13	0.0095	1415	13	0.0092	1465	13	0.0089
1366	13	0.0095	1416	13	0.0092	1466	14	0.0095
1367	14	0.0102	1417	13	0.0092	1467	13	0.0089
1368	13	0.0095	1418	13	0.0092	1468	13	0.0089
1369	13	0.0095	1419	13	0.0092	1469	14	0.0095
1370	13	0.0095	1420	13	0.0092	1470	13	0.0088
1371	13	0.0095	1421	13	0.0091	1471	14	0.0095
1372	13	0.0095	1422	13	0.0091	1472	12	0.0082
1373	14	0.0102	1423	14	0.0098	1473	13	0.0088
1374	13	0.0095	1424	13	0.0091	1474	13	0.0088
1375	14	0.0102	1425	14	0.0098	1475	13	0.0088
1376	12	0.0087	1426	13	0.0091	1476	13	0.0088
1377	13	0.0094	1427	14	0.0098	1477	13	0.0088
1378	13	0.0094	1428	13	0.0091	1478	13	0.0088
1379	13	0.0094	1429	14	0.0098	1479	14	0.0095
1380	13	0.0094	1430	13	0.0091	1480	13	0.0088
1381	13	0.0094	1431	14	0.0098	1481	13	0.0088
1382	13	0.0094	1432	13	0.0091	1482	13	0.0088
1383	14	0.0101	1433	14	0.0098	1483	14	0.0094
1384	13	0.0094	1434	13	0.0091	1484	13	0.0088
1385	13	0.0094	1435	14	0.0098	1485	13	0.0088
1386	13	0.0094	1436	13	0.0091	1486	13	0.0087
1387	13	0.0094	1437	14	0.0097	1487	14	0.0094
1388	13	0.0094	1438	14	0.0097	1488	13	0.0087
1389	14	0.0101	1439	14	0.0097	1489	14	0.0094
1390	14	0.0101	1440	12	0.0083	1490	13	0.0087
1391	14	0.0101	1441	13	0.0090	1491	14	0.0094
1392	13	0.0093	1442	13	0.0090	1492	13	0.0087
1393	13	0.0093	1443	13	0.0090	1493	14	0.0094
1394	13	0.0093	1444	13	0.0090	1494	13	0.0087
1395	14	0.0100	1445	13	0.0090	1495	13	0.0087
1396	13	0.0093	1446	13	0.0090	1496	13	0.0087
1397	14	0.0100	1447	14	0.0097	1497	14	0.0094
1398	13	0.0093	1448	13	0.0090	1498	13	0.0087
1399	13	0.0093	1449	13	0.0090	1499	14	0.0093
1400	13	0.0093	1450	13	0.0090	1500	13	0.0087

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
1501	14	0.0093	1551	13	0.0084	1601	13	0.0081
1502	14	0.0093	1552	12	0.0077	1602	13	0.0081
1503	14	0.0093	1553	13	0.0084	1603	13	0.0081
1504	13	0.0086	1554	13	0.0084	1604	13	0.0081
1505	14	0.0093	1555	13	0.0084	1605	13	0.0081
1506	14	0.0093	1556	13	0.0084	1606	13	0.0081
1507	14	0.0093	1557	14	0.0090	1607	14	0.0087
1508	13	0.0086	1558	13	0.0083	1608	13	0.0081
1509	14	0.0093	1559	14	0.0090	1609	[illegible]	0.0081
1510	14	0.0093	1560	12	0.0077	1610	13	0.0081
1511	14	0.0093	1561	13	0.0083	1611	14	0.0087
1512	13	0.0086	1562	13	0.0083	1612	13	0.0081
1513	14	0.0093	1563	13	0.0083	1613	13	0.0081
1514	14	0.0092	1564	13	0.0083	1614	14	0.0087
1515	14	0.0092	1565	14	0.0089	1615	13	0.0080
1516	14	0.0092	1566	13	0.0083	1616	13	0.0080
1517	14	0.0092	1567	14	0.0089	1617	13	0.0080
1518	13	0.0086	1568	12	0.0077	1618	13	0.0080
1519	14	0.0092	1569	13	0.0083	1619	14	0.0086
1520	13	0.0086	1570	13	0.0083	1620	13	0.0080
1521	14	0.0092	1571	13	0.0083	1621	14	0.0086
1522	14	0.0092	1572	13	0.0083	1622	14	0.0086
1523	14	0.0092	1573	14	0.0089	1623	14	0.0086
1524	13	0.0085	1574	13	0.0083	1624	13	0.0080
1525	14	0.0092	1575	14	0.0089	1625	13	0.0080
1526	14	0.0092	1576	13	0.0082	1626	13	0.0080
1527	14	0.0092	1577	14	0.0089	1627	14	0.0086
1528	13	0.0085	1578	14	0.0089	1628	14	0.0086
1529	14	0.0092	1579	14	0.0089	1629	14	0.0086
1530	13	0.0085	1580	13	0.0082	1630	13	0.0080
1531	14	0.0091	1581	14	0.0089	1631	14	0.0086
1532	14	0.0091	1582	14	0.0088	1632	12	0.0074
1533	14	0.0091	1583	14	0.0088	1633	13	0.0080
1534	14	0.0091	1584	12	0.0076	1634	13	0.0080
1535	14	0.0091	1585	13	0.0082	1635	13	0.0080
1536	11	0.0072	1586	13	0.0082	1636	13	0.0079
1537	12	0.0078	1587	13	0.0082	1637	14	0.0086
1538	12	0.0078	1588	13	0.0082	1638	13	0.0079
1539	12	0.0078	1589	14	0.0088	1639	14	0.0085
1540	12	0.0078	1590	13	0.0082	1640	13	0.0079
1541	13	0.0084	1591	14	0.0088	1641	13	0.0079
1542	12	0.0078	1592	13	0.0082	1642	14	0.0085
1543	13	0.0084	1593	13	0.0082	1643	14	0.0085
1544	12	0.0078	1594	14	0.0088	1644	13	0.0079
1545	13	0.0084	1595	14	0.0088	1645	14	0.0085
1546	13	0.0084	1596	13	0.0081	1646	14	0.0085
1547	13	0.0084	1597	14	0.0088	1647	14	0.0085
1548	12	0.0078	1598	14	0.0088	1648	13	0.0079
1549	13	0.0084	1599	14	0.0088	1649	13	0.0079
1550	13	0.0084	1600	12	0.0075	1650	13	0.0079

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
1651	14	0.0085	1701	14	0.0082	1751	14	0.0080
1652	13	0.0079	1702	14	0.0082	1752	13	0.0074
1653	14	0.0085	1703	14	0.0082	1753	14	0.0080
1654	14	0.0085	1704	13	0.0076	1754	14	0.0080
1655	14	0.0085	1705	14	0.0082	1755	13	0.0074
1656	13	0.0079	1706	14	0.0082	1756	14	0.0080
1657	14	0.0084	1707	14	0.0082	1757	14	0.0080
1658	14	0.0084	1708	14	0.0082	1758	13	0.0074
1659	13	0.0078	1709	14	0.0082	1759	14	0.0080
1660	13	0.0078	1710	14	0.0082	1760	13	0.0074
1661	14	0.0084	1711	14	0.0082	1761	14	0.0080
1662	14	0.0084	1712	13	0.0076	1762	14	0.0079
1663	14	0.0084	1713	14	0.0082	1763	14	0.0079
1664	12	0.0072	1714	14	0.0082	1764	13	0.0074
1665	13	0.0078	1715	14	0.0082	1765	14	0.0079
1666	13	0.0078	1716	13	0.0076	1766	14	0.0079
1667	13	0.0078	1717	14	0.0082	1767	14	0.0079
1668	13	0.0078	1718	14	0.0081	1768	13	0.0074
1669	13	0.0078	1719	14	0.0081	1769	14	0.0079
1670	13	0.0078	1720	13	0.0076	1770	14	0.0079
1671	13	0.0078	1721	14	0.0081	1771	14	0.0079
1672	13	0.0078	1722	14	0.0081	1772	14	0.0079
1673	13	0.0078	1723	14	0.0081	1773	14	0.0079
1674	13	0.0078	1724	14	0.0081	1774	14	0.0079
1675	14	0.0084	1725	14	0.0081	1775	14	0.0079
1676	13	0.0078	1726	14	0.0081	1776	13	0.0073
1677	13	0.0078	1727	14	0.0081	1777	14	0.0079
1678	13	0.0077	1728	12	0.0069	1778	14	0.0079
1679	14	0.0083	1729	13	0.0075	1779	14	0.0079
1680	13	0.0077	1730	13	0.0075	1780	14	0.0079
1681	13	0.0077	1731	13	0.0075	1781	14	0.0079
1682	13	0.0077	1732	13	0.0075	1782	13	0.0073
1683	13	0.0077	1733	13	0.0075	1783	14	0.0079
1684	13	0.0077	1734	13	0.0075	1784	14	0.0078
1685	14	0.0083	1735	14	0.0081	1785	14	0.0078
1686	13	0.0077	1736	13	0.0075	1786	14	0.0078
1687	14	0.0083	1737	13	0.0075	1787	14	0.0078
1688	13	0.0077	1738	13	0.0075	1788	13	0.0073
1689	14	0.0083	1739	14	0.0081	1789	14	0.0078
1690	13	0.0077	1740	13	0.0075	1790	14	0.0078
1691	14	0.0083	1741	14	0.0080	1791	14	0.0078
1692	13	0.0077	1742	14	0.0080	1792	12	0.0067
1693	14	0.0083	1743	13	0.0075	1793	13	0.0073
1694	14	0.0083	1744	13	0.0075	1794	13	0.0072
1695	14	0.0083	1745	14	0.0080	1795	13	0.0072
1696	13	0.0077	1746	13	0.0074	1796	13	0.0072
1697	14	0.0082	1747	14	0.0080	1797	13	0.0072
1698	13	0.0077	1748	13	0.0074	1798	13	0.0072
1699	14	0.0082	1749	14	0.0080	1799	13	0.0072
1700	13	0.0076	1750	14	0.0080	1800	13	0.0072

N	L(N)	L(N)/N	N	L(N)	L(N)/N	N	L(N)	L(N)/N
1801	13	0.0072	1851	14	0.0076	1901	14	0.0074
1802	13	0.0072	1852	14	0.0076	1902	14	0.0074
1803	14	0.0078	1853	14	0.0076	1903	15	0.0079
1804	13	0.0072	1854	14	0.0076	1904	13	0.0068
1805	14	0.0078	1855	14	0.0075	1905	14	0.0073
1806	13	0.0072	1856	13	0.0070	1906	14	0.0073
1807	14	0.0077	1857	14	0.0075	1907	14	0.0073
1808	13	0.0072	1858	14	0.0075	1908	14	0.0073
1809	14	0.0077	1859	14	0.0075	1909	14	0.0073
1810	13	0.0072	1860	14	0.0075	1910	14	0.0073
1811	14	0.0077	1861	14	0.0075	1911	14	0.0073
1812	13	0.0072	1862	14	0.0075	1912	14	0.0073
1813	14	0.0077	1863	14	0.0075	1913	14	0.0073
1814	14	0.0077	1864	13	0.0070	1914	14	0.0073
1815	14	0.0077	1865	14	0.0075	1915	14	0.0073
1816	13	0.0072	1866	14	0.0075	1916	14	0.0073
1817	14	0.0077	1867	14	0.0075	1917	15	0.0078
1818	14	0.0077	1868	14	0.0075	1918	14	0.0073
1819	14	0.0077	1869	14	0.0075	1919	14	0.0073
1820	13	0.0071	1870	14	0.0075	1920	12	0.0062
1821	14	0.0077	1871	14	0.0075	1921	13	0.0068
1822	14	0.0077	1872	13	0.0069	1922	13	0.0068
1823	14	0.0077	1873	14	0.0075	1923	13	0.0068
1824	13	0.0071	1874	14	0.0075	1924	13	0.0068
1825	14	0.0077	1875	14	0.0075	1925	13	0.0068
1826	14	0.0077	1876	14	0.0075	1926	13	0.0067
1827	14	0.0077	1877	14	0.0075	1927	14	0.0073
1828	13	0.0071	1878	14	0.0075	1928	13	0.0067
1829	14	0.0077	1879	14	0.0075	1929	13	0.0067
1830	14	0.0077	1880	14	0.0074	1930	13	0.0067
1831	14	0.0076	1881	14	0.0074	1931	14	0.0073
1832	13	0.0071	1882	14	0.0074	1932	13	0.0067
1833	14	0.0076	1883	14	0.0074	1933	14	0.0072
1834	14	0.0076	1884	14	0.0074	1934	14	0.0072
1835	14	0.0076	1885	14	0.0074	1935	13	0.0067
1836	13	0.0071	1886	14	0.0074	1936	13	0.0067
1837	14	0.0076	1887	14	0.0074	1937	14	0.0072
1838	14	0.0076	1888	13	0.0069	1938	13	0.0067
1839	14	0.0076	1889	14	0.0074	1939	14	0.0072
1840	13	0.0071	1890	14	0.0074	1940	13	0.0067
1841	14	0.0076	1891	14	0.0074	1941	14	0.0072
1842	14	0.0076	1892	14	0.0074	1942	14	0.0072
1843	14	0.0076	1893	14	0.0074	1943	14	0.0072
1844	14	0.0076	1894	14	0.0074	1944	13	0.0067
1845	14	0.0076	1895	14	0.0074	1945	14	0.0072
1846	14	0.0076	1896	14	0.0074	1946	14	0.0072
1847	14	0.0076	1897	14	0.0074	1947	14	0.0072
1848	13	0.0070	1898	14	0.0074	1948	14	0.0072
1849	14	0.0076	1899	14	0.0074	1949	14	0.0072
1850	14	0.0076	1900	14	0.0074	1950	13	0.0067

APPENDIX II

$$\ell(n) = \ell(2n)$$

2n	2n	2n	2n
382	5902*	12982	15986
1402	8562	13406	16086
1486*	8846	13502	16166
2222	8982	14494	16538
2778	9486	14638	16850
2958*	10674	14926	17074
4206	11034	14962	17678
4430	11790*	15142	17706
4750	12638	15502	17942
5362	12734	15818	

* $2n = 2^{m+1}(23) + 14, \quad m \geq 5$